**Masters and Ph.D. Qualifying Exam** **Analysis: Math 825/826**

Wednesday, January 27, 2017, 2:30 – 8:30p.m.

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• Work 5 out of 6 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.
• Do not interpret a problem in such a way that it becomes trivial.

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(1) Let $(X, \rho)$ be a metric space.
 (a) State the definition for a compact set $K \subseteq X$.
 (b) Suppose that $\{x_j\}_{j=1}^{\infty}$ is a sequence in $X$ that converges to $x \in X$. Verify that the set $K := \{x\} \cup \{x_j : j \in \mathbb{N}\}$ is compact.

(2) Define $f : [0,1] \to [-1,1]$ by

$$f(x) := \begin{cases} x \sin\left(\frac{1}{x}\right), & 0 < x \leq 1; \\ 0, & x = 0. \end{cases}$$

 (a) Determine, with justification, whether $f$ is of bounded variation on the interval $[0,1]$.
 (b) Determine, with justification, whether $f$ is continuous on the interval $[0,1]$.

(3) Let $f : [0,1] \to \mathbb{R}$ be a non-decreasing function. Prove that $\lim_{x \to c^-} f(x)$ exists for each $c \in (0,1]$ and that $\lim_{x \to c^+} f(x)$ exists for each $c \in [0,1)$.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. Let $c \in \mathbb{R}$ be given, and suppose that $f$ has the following property: there is an $L \in \mathbb{R}$ such that for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left| \frac{f(r) - f(c)}{r - c} - L \right| < \varepsilon, \qquad \text{whenever } r \in \mathbb{Q} \text{ and } 0 < |r - c| < \delta.$$

Prove that $f$ is differentiable at $c$ and that $f'(c) = L$.

(5) Let $f : (0,1) \to \mathbb{R}$ be a continuous function. Fix $c \in (0,1)$, and suppose that $f$ is differentiable on $(0,1)\backslash\{c\}$. Assuming that $\lim_{x \to c} f'(x)$ exists, prove that $f$ is differentiable at $c$.

(6) (a) Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of real numbers such that $\sum_{j=1}^{\infty} x_j$ converges, but $\sum_{j=1}^{\infty} x_j^2$ diverges. Argue that $\sum_{j=1}^{\infty} x_j$ must converge conditionally.
 (b) Let $\{x_j\}_{j=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$ be sequences of real numbers such that $\sum_{j=1}^{\infty} x_j$ and $\sum_{j=1}^{\infty} y_j$ are both absolutely convergent. Prove that the series $\sum_{j=1}^{\infty} \sqrt{|x_j y_j|}$ is also absolutely convergent.