Masters and Ph.D. Qualifying Exam **Analysis: Math 825/826**

Wednesday, January 23, 2019.

• Work 5 out of 6 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.

• Do not interpret a problem in such a way that it becomes trivial.

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(1) (a) Let $f_n(x) = \dfrac{1}{1+n^2x^2}$ and $g_n(x) = nx(1-x)^n$ for $x \in [0,1]$. Prove that $\{f_n\}$ and $\{g_n\}$ converge pointwise but not uniformly on $[0,1]$.

(b) Are the families $\{f_n\}$, respectively $\{g_n\}$ given in part (a) equicontinuous? Clearly motivate your answer.

(2) Consider the following subset of the metric space $(\mathcal{C}_\mathrm{b}([0,1]), \rho_\infty)$:

$$A := \big\{ f \in \mathcal{C}_\mathrm{b}([0,1]) : f([0,1]) \subseteq [0,1] \big\}.$$

(a) Determine whether A is bounded, and if so what is its diameter.

(b) Determine whether A is closed in $\mathcal{C}_\mathrm{b}([0,1])$.

(c) Determine whether A is compact in $\mathcal{C}_\mathrm{b}([0,1])$.

(3) Determine the values of $x \in \mathbb{R}$ for which

$$\sum_{n=1}^{\infty} \frac{x^n}{1+n|x|^n}$$

converges.

(4) Suppose that $f : [0,1] \to \mathbb{R}$ is differentiable and $f(0) = 0$. Assume that there is a $k > 0$ such that

$$|f'(x)| \le k|f(x)|$$

for all $x \in [0,1]$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

(5) Use the Riemann condition to show that $f \in \mathcal{R}_\alpha[0,4]$ where $f(x) = e^{2x}$ and

$$\alpha(x) = \begin{cases} x+1, & 0 \le x \le 2 \\ 3x-2, & 2 < x \le 4. \end{cases}$$

Compute the value of the Riemann-Stieltjes integral $\displaystyle\int_0^4 f(x)d\alpha$.

(6) Use the Heine-Borel Theorem to prove that if f is continuous on $[a,b]$ and $f(x) > 0$ for every $x \in [a,b]$ then there exists $\varepsilon > 0$ such that $f(x) \ge \varepsilon$ for every $x \in [a,b]$.