Masters and Ph.D. Qualifying Exam **Analysis: Math 825/826**

Thursday, May 28, 2019

• Work 5 out of 6 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.
• Do not interpret a problem in such a way that it becomes trivial.

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(1) Let (M, d_M) and (N, d_N) be metric spaces. Define $d_{M\times N} : (M \times N) \times (M \times N) \to \mathbb{R}$ by

$$d_{M\times N}((x_1, y_1), (x_2, y_2)) := d_M(x_1, x_2) + d_N(y_1, y_2), \quad \text{for all } x_1, x_2 \in M \text{ and } y_1, y_2 \in N.$$

(a) Prove that $(M \times N, d_{M\times N})$ is a metric space.

(b) Let $S \subseteq M$ and $T \subseteq N$ be compact sets in (M, d_M) and (N, d_N), respectively. Prove that $S \times T$ is a compact set in $(M \times N, d_{M\times N})$.

(2) Let $\{a_n\}_{n=1}^{\infty} \subset (0, \infty)$ be given, and assume that $\sum_{n=1}^{\infty} a_n$ converges.

(a) Show that

$$\sum_{n=1}^{\infty} \frac{a_n}{1 + a_n} \text{ converges} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{1 + a_n} \text{ diverges.}$$

(b) Suppose that $\{b_n\}_{n=1}^{\infty} \subset \mathbb{R}$ satisfies $|b_{n+1} - b_n| \le a_n$, for every $n \in \mathbb{N}$. Prove $\{b_n\}_{n=1}^{\infty}$ is convergent.

(3) Let $f : [0, \infty) \to \mathbb{R}$ be continuous on $[0, \infty)$, and define $F : (0, \infty) \to \mathbb{R}$ by

$$F(x) := \sup_{0 \le s < x} f(s) = \operatorname{lub} \{f(s) : 0 \le s < x\}.$$

(a) Show that $\lim_{x\to 0^+} F(x)$ exists.

(b) Prove that F is continuous on $(0, \infty)$.

(c) Suppose that f is bounded, as well as continuous, on $[0, \infty)$. Prove that F is uniformly continuous on $[0, \infty)$.

(4) Consider the formal series

$$\sum_{n=0}^{\infty} \mathrm{e}^{-2nx} \left(1 - \mathrm{e}^{-x}\right).$$

Prove or disprove each of the following statements.

(a) The series converges uniformly on the interval $[1, \infty)$.

(b) The series converges uniformly on the interval $[0, 1]$.

(5) (a) Let $\alpha : [0, 1] \to \mathbb{R}$ be an increasing function that is continuous at 0, and let $f : [0, 1] \to \mathbb{R}$ be a bounded function. Suppose that, for each $0 < c < 1$, the function f is Riemann-Stieltjes integrable with respect to α over $[c, 1]$. Prove that f is Riemann-Stieljes integrable with respect to α over $[0, 1]$.

(b) Let $\{q_n\}_{n=1}^{\infty} \subset [0, 1]$ be an enumeration of $\mathbb{Q} \cap [0, 1]$, the rational numbers in $[0, 1]$. Define $\beta : [0, 1] \to \mathbb{R}$ by

$$\beta(x) := \begin{cases} \dfrac{1}{n}, & x = q_n, \\ 0, & x \notin \mathbb{Q} \cap [0, 1]. \end{cases}$$

Prove or disprove: β has bounded variation on $[0, 1]$.

(6) Suppose that $f : [0, 2] \to \mathbb{R}$ has the following properties:

- f is continuous on $[0, 2]$ and differentiable on $(0, 2)$;
- $f(0) = f(2) = 0$;
- there exists $x_0 \in (0, 2)$ such that $|f(x_0)| \ge 1$.

(a) Prove that there exists $c \in (0, 2)$ such that $|f'(c)| \ge 1$.

(b) There exists $k > 1$ such that for each $\gamma \in [0, k]$, there exists some $c_\gamma \in (0, 2)$ such that $|f'(c_\gamma)| = \gamma$.