Masters and Ph.D. Qualifying Exam **Analysis: Math 825/826**

Thursday, June 1, 2023

• Work 5 of 6 problems. • Each problem is worth 20 points. • Use one side of the paper only and hand your work in order.

• Do not interpret a problem in such a way that it becomes trivial.

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(1) Let $M < \infty$, $0 < \beta \leq 1$, and $\infty < a < b < \infty$ be given. Suppose that $\{f_j\}_{j=1}^{\infty} \subseteq \mathcal{F}$, where $\mathcal{F} = \left\{ f \in \mathcal{C}([a,b]) : |f(a)| \leq M \text{ and } \sup_{x,y\in[a,b]} |f(x) - f(y)| \leq M|x-y|^{\beta} \right\}$.

(a) Show that $\{f_j\}_{j=1}^{\infty}$ has a subsequence that converges uniformly to some $f_0 \in \mathcal{C}([a,b])$.

(b) Determine whether the function f_0 identified in part (a) must belong to $\mathcal{F}$. (Justify your claim.)

(2) Let $\alpha \in \mathcal{C}(\mathbb{R})$ be a continuously differentiable function satisfying $\alpha(0) = 0$ and $\alpha'(x) > 0$ at each $x \in \mathbb{R}$. For each $j \in \mathbb{N}$, define the functions $f_j, g_j \in \mathcal{C}([0,1])$ by

$$f_j(x) = x + \tfrac{1}{j} \qquad \text{and} \qquad g_j(x) = \frac{j\alpha(x)}{1 + j\alpha(x)}.$$

Evaluate $\displaystyle\lim_{j\to\infty} \int_0^1 f_j(x)\,\mathrm{d}g_j(x)$. (Carefully justify your claim.)

(3) Suppose that $f \in \mathcal{C}([0,2])$ satisfies $f(0) = f(2) = 0$ and $f(x) > x(2-x)$ on $(0,2)$.

(a) Prove that the series $\displaystyle\sum_{j=0}^{\infty} (1-x)^{2j} f(x)$ converges pointwise **but not** uniformly on $[0,2]$.

(b) Prove that the series $\displaystyle\sum_{j=0}^{\infty} (-1)^j \frac{(1-x)^{2j} f(x)}{j+1}$ converges uniformly on $[0,2]$.

(4) Let X be a vector space, and For each $j \in \mathbb{N}$, suppose that $\|\cdot\|_j$ is a norm on X. Define

$$X_0 = \{x \in X : \|x\|_{X_0} < \infty\} \subseteq X, \qquad \text{where} \qquad \|x\|_{X_0} = \sum_{j=1}^{\infty} \|x\|_j.$$

(a) Verify $(X_0, \|\cdot\|_{X_0})$ is a normed vector space.

(b) Assume that $E \subseteq X_0$ is compact in $(X_0, \|\cdot\|_{X_0})$. For each $j \in \mathbb{N}$, prove that E must also be compact in $(X, \|\cdot\|_j)$.

(c) For each $j \in \mathbb{N}$, suppose both of the following hold:

- $(X, \|\cdot\|_j)$ is complete
- $\forall x \in X, \quad \|x\|_{j+1} \leq \left(\frac{j}{j+1}\right)^2 \|x\|_j.$

Prove that $(X_0, \|\cdot\|_{X_0})$ is complete.

(5) With (X,d) a metric space, let $\mathcal{F} \subseteq \mathcal{C}(X)$ be a given. ($\mathcal{F}$ is not necessarily countable.) Assume that $\sup_{f\in\mathcal{F}} |f(x)| < \infty$, for each $x \in X$.

(a) Argue that for each $x \in X$, there is a unique value for $g(x) = \sup_{f\in\mathcal{F}} f(x)$ in $\mathbb{R}$.

(b) Let $g : X \to \mathbb{R}$ be as in part (a). For each $x \in X$, prove that $\liminf_{y\to x} g(y) \geq g(x)$.

(6) Set $I = (0,\infty)$. Provide a counterexample or a proof for each of the following statements. (Carefully justify your claims.)

(a) If $f \in \mathcal{C}(I)$ is differentiable on I and $\lim_{x\to\infty} f(x)$ exists in $\mathbb{R}$, then $\lim_{x\to\infty} f'(x) = 0$.

(b) If $f \in \mathcal{C}(I)$ is differentiable on I, $\lim_{x\to\infty} f'(x)$ exists in $\mathbb{R}$, and $\sup_{x\in\mathbb{R}} |f(x)| < \infty$, then $\lim_{x\to\infty} f'(x) = 0$.

(c) If $f \in \mathcal{C}(I)$ and $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to\infty} f(x)$ both exist in $\mathbb{R}$, then f is uniformly continuous on I.