Masters and Ph.D. Qualifying Exam **Analysis: Math 825/826**

Wednesday, January 18, 2023.

• Work 5 out of 6 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.

• Do not interpret a problem in such a way that it becomes trivial.

(1) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0 \in \mathbb{R}$. Let $\{a_n\}_{n=1}^{\infty} \subseteq (-\infty, x_0)$ and $\{b_n\}_{n=1}^{\infty} \subseteq (x_0, \infty)$ be sequences that both converge to x_0. Prove that

$$\lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{b_n - a_n} = f'(x_0).$$

(You may assume that $x_0 = 0$).

(2) Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset (0, \infty)$ be given.

(a) Assume that $\limsup_{n\to\infty}(a_n/b_n) < \infty$. Prove there exists $M \in \mathbb{R}$ such that $a_n \leq M b_n$, for all $n \in \mathbb{N}$.

(b) Suppose the sequence $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ converges in $\mathbb{R}$. Must $\limsup\limits_{n\to\infty}(a_n/b_n)$ be finite?

(3) (a) Suppose that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence from $(\mathcal{C}([0,1]), \rho_\infty)$. Determine whether $\{f_n\}_{n=1}^{\infty}$ must be uniformly equicontinuous.

(b) Suppose that $\mathcal{F} \subseteq (\mathcal{C}([0,1]), \rho_\infty)$ is closed and bounded but not uniformly equicontinuous. Prove that $\mathcal{F}$ is not compact in $(\mathcal{C}([0,1]), \rho_\infty)$.

(4) Use the Riemann condition to show that $f \in \mathcal{R}_\alpha[0,2]$ where $f(x) = \tan(\pi x/4)$ and

$$\alpha(x) = \begin{cases} x+1, & 0 \leq x \leq 1 \\ 4x, & 1 < x \leq 2. \end{cases}$$

Compute the value of the Riemann-Stieltjes integral $\displaystyle\int_0^2 f(x)d\alpha$.

(5) Determine all the values of $x \in \mathbb{R}$ for which the series below converges

$$\sum_{n=1}^{\infty} \frac{x^n}{1 + n|x|^n}.$$

(6) Let (M, d) be a metric space. Define the functions $d', d'' : M \times M \to [0, \infty)$ by

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}, \quad \text{and} \quad d''(x,y) = \min\{1, d(x,y)\}.$$

(a) Show that d', d'' are both metrics.

(b) Show that d, d', d'' are equivalent metrics on the space M.