Mathematics 842-843
Masters/Qualifying Exam Jan 1997

1. Consider the convection-diffusion equation

$$\frac{\partial}{\partial t}(u - u^2/2) = u_{xx} - u_x$$

(a) What is the speed of right traveling waves of the form $u = U(x - ct)$ for this PDE that satisfies the conditions $U(-\infty) = 1,\ \ U(\infty) = 0$?

(b) Determine explicitly the traveling wave that also satisfies $U(0) = 0.5$ and sketch a graph of it.

2. Find a three-term asymptotic approximation to the function

$$I(x) = \int_0^x e^{-s^2} dx$$

as $x \to 0$.

3. In the quarter plane $x, y > 0$, where the temperature is initially zero, heat flows only in the y direction; along the edge $y = 0$ heat is convected along the x–axis, and the temperature is constantly one at the point $x = y = 0$. The boundary value problem for the temperature $u(x, y, t)$ is

$$\begin{aligned} u_t &= u_{yy}, \ \ x, t, y > 0 \\ u(x, y, 0) &= 0, \ \ x, y > 0 \\ u(0, 0, t) &= 1, \ \ t > 0 \\ u_t(x, 0, t) + u_x(x, 0, t) &= 0, \ \ x, t > 0 \end{aligned}$$

Find a bounded solution to this problem using Laplace transforms. You may use the enclosed table.

4. In this problem use the notation $||u|| = (\int_0^1 u(x)^2 dx)^{1/2}$. Assume that

$$\lambda = \min\{\frac{||u'||^2}{||u||^2} \ : \ u \in C^2[0,1], \ u(0) = u(1) = 0\}$$

exists. Prove that λ is the smallest eigenvalue of the problem

$$\phi'' + \lambda\phi = 0, \ \ 0 < x < 1, \ \ \phi(0) = \phi(1) = 0$$

.

5. Consider the problem

$$\begin{aligned} \epsilon u'' + b(t)u' + u &= 0, \ \ 0 < t < T; \ \ 0 < \epsilon \ll 1 \\ u(0) = 0, \ \ u'(0) &= \beta/\epsilon + \gamma \end{aligned}$$

where b is smooth and $b(t) > 0$, and $T, \beta, \gamma > 0$. Find a uniformly valid, leading order approximation to this problem.