Applied Math, January 2002

Do three of the first four and three of the second four problems.

1. Consider a very long rod ($x > 0$) that is laterally insulated and where heat is flowing in the x-direction; the temperature $u = u(x,t)$ is governed by the one-dimensional diffusion (heat) equation. The diffusivity is $k = 0.007$ $\text{cm}^2\text{sec}^{-1}$. The initial temperature of the rod is 7000 degrees F, and the boundary $x = 0$ is maintained at zero degrees for all time $t > 0$. If at some fixed time τ we measure the gradient at $x = 0$ and find $u_x(0,\tau) = 3.7(10)^{-4}$ deg cm^{-1}, what is τ?

2. Consider the following reaction-convection-diffusion equation for $u(x,t)$:
$$\frac{\partial}{\partial t}\left((1+b)u - au^2\right) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}, \tag{1}$$
where $0 < a < b$. Show that there exists a traveling wave solution to (1) of the form $u = U(z)$, $z = x - ct$, for some wave speed $c > 0$, where the wave shape $U = U(z)$ satisfies the boundary conditions
$$U(-\infty) = 1, \quad U(+\infty) = 0.$$

3. A fluid of constant density ρ_0 is flowing through a tube of length L that has a variable cross-sectional area $A(x)$, $0 \leq x \leq L$. The variation in $A(x)$ is small so that the flow can be considered one dimensional. Let $u = u(x,t)$ denote the (Eulerian) velocity of the fluid. Derive, from first principles, a mass conservation law for the fluid motion and simplify it as much as possible.

4. Consider the system of equations for $u = u(x,t)$ and $v = v(x,t)$ on the domain **D**: $0 < x < 1$, $t > 0$:
$$\begin{aligned}\frac{\partial u}{\partial t} &= -\frac{\partial u}{\partial x} - au,\\ \frac{\partial v}{\partial t} &= -\frac{\partial v}{\partial x} + u,\end{aligned}$$
where $a > 0$ is a constant. Initial and boundary conditions are given by
$$\begin{aligned}u(x,0) &= v(x,0) = 0, \quad 0 \leq x \leq 1,\\ u(0,t) &= b, \quad t > 0,\end{aligned}$$
where $b > 0$ is a constant. Find analytic formulas for the solution in the domain **D**.

5. Let P be the power required to keep a ship of length L moving at a constant speed V. Suppose that P depends on the density ρ of the water, the acceleration of gravity g, the viscosity of the water ν (with dimension length^2/time) as well as on V and L. Show that for some function f,

$$P = \rho L^2 V^3 f(\text{Fr},\text{Re}),$$

where Fr and Re are the Froude and Reynolds numbers:

$$\text{F}r = \frac{V}{\sqrt{Lg}}, \quad \text{R}e = \frac{VL}{\nu}.$$

6. *The complementary error function* is

$$erfc(\lambda) = \frac{2}{\sqrt{\pi}} \int_\lambda^\infty e^{-t^2}\, dt.$$

Show that to leading order,

$$erfc(\lambda) \sim \frac{1}{\sqrt{\pi}} \frac{e^{-\lambda^2}}{\lambda} \quad \text{as} \quad \lambda \to \infty.$$

7. Consider the two-point boundary value problem

$$(P) \begin{cases} y'' + 4y = f(x) & \text{for } 0 < x < \pi, \\ y'(0) = 0, \\ y(\pi) = 0, \end{cases}$$

where f is continuous on $[0, \pi]$. Recall that the Green's function G, if it exists, gives an integral representation of the solution y:

$$y(x) = \int_0^\pi G(x, \xi) f(\xi)\, d\xi.$$

Is there a Green's function for (P)? If there is, find it. If there isn't, explain how you know.

8. A mass m moves in the xy-plane subject to a central force field with potential

$$V = -\frac{k}{\sqrt{x^2 + y^2}},$$

where $k > 0$ is a constant. Show that the Lagranigian in polar coordinates is

$$L(r, \theta) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}.$$

Use Hamilton's principle to derive the equations of motion.