Applied Math (842-843) Qualifying Exam **January 2003**

Instructions: You are to work five problems of the six problems below. If you attempt more than five problems, clearly indicate which you wish to be graded. All problems have equal value. If you have questions about the wording of a particular problem, ask for clarification. In no case should you interpret a problem in such a way that its solution becomes trivial. Time: 3 hours.

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1. Find a leading order approximation, uniformly valid on $[0,1]$ as $\epsilon \to 0$, to the solution of the two point boundary value problem

$$\epsilon \frac{d^2y}{dt^2} + (1+\epsilon)\frac{dy}{dt} + y^2 = 0,\ y(0) = 0,\ y(1) = 1/2,\ 0 < \epsilon \ll 1.$$

2. Find the extremal(s) of the functional

$$J(y) = \int_0^1 \left(y^2 + (y')^2\right) dx$$

with domain $A = \{y(x) \in C^2[0,1] \,|\, y(1) = 1\}$. Could these extremals could be maxima or minima? Explain.

3. A new species of prey is introduced at time $T = 0$ into an environment where its single predator has existed with a constant population. Let X and Y be the prey, predator populations, respectively and assume the dynamics of the populations are governed by the system

$$\begin{aligned} \frac{dX}{dT} &= (K - AX - BY)X \\ \frac{dY}{dT} &= CXY, \end{aligned}$$

where K, A, B, C are positive constants. Determine the long term behavior of the populations by phase plane analysis. Are there any periodic solutions? Justify your answer.

4. Consider the boundary value problem

$$(P)\ \begin{cases} u'' + 2u' + u = f(x) \\ u(0) = u(1) = 0. \end{cases}$$

Either find the Green's function for (P), or explain how you can be sure that it doesn't have one.

5. Some material (call it "stuff") has concentration $u(x,t)$ at the point $x \in \mathbb{R}^3$ and time $t \geq 0$. Let Φ be the flux density vector for the flow of stuff, and f the net rate per unit volume at which stuff is created at (x,t).

(a) Derive the balance equation

$$u_t + \nabla \cdot \Phi = f. \tag{1}$$

Assume anything you want about the regularity of u, Φ and f.
(b) Suppose that stuff moves by diffusion alone. Give a brief, informal justification for the constitutive assumption

$$\Phi = -D\nabla u \tag{2}$$

for a constant $D > 0$.
(c) Suppose that the stuff is confined to a bounded volume $V \subset \mathbb{R}^3$ with smooth, closed boundary ∂V. Use this information along with (1) and (2) to derive the initial-boundary value problem

$$(IBVP) \begin{cases} u_t &= D\Delta u + f, & \text{for } x \in V \text{ and } t > 0, \\ \nabla u \cdot \mathbf{n} &= 0, & \text{for } x \in \partial V, \\ u(x,0) &= g(x), & \text{for } x \in V, \end{cases}$$

where $\mathbf{n}$ is the outer unit normal to ∂V and g is the initial density of stuff in V.
(d) Suppose that the source term f depends only on x and t. Show that $(IBVP)$ has at most one smooth solution.

6. At time t^* and point x^* the solution to the pure initial value problem

$$(IVP) \begin{cases} u_t + F(u)_x &= 0, & \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ u(x,0) &= 0, & \text{for } x \in \mathbb{R} \end{cases}$$

develops a discontinuity. For $t > t^*$, let $[u]$ and $[F(u)]$ be respectively the jumps in u and $[F(u)]$ across the discontinuity and $s = s(t)$ the speed of the discontinuity. Derive the jump condition

$$[F(u)] = \frac{ds}{dt}\,[u]\,.$$