Mathematics 842-843 Masters/Qualifying Exam
June 1996

Work all the problems–term is three hours.

1. Find a one-term perturbation approximation to the boundary value problem

$$\begin{aligned} \epsilon u'' + (x+1)u' + u &= 2x, \quad 0 < x < 1, \quad 0 < \epsilon \ll 1 \\ u(0) &= 1, \quad u(1) = 2 \end{aligned}$$

2. Consider the functional defined by

$$J(u) = \int_a^b p(x)^2 u'^2 dx, \quad u(a) = A, \quad u(b) = B; \quad p \in C[a,b], \quad p > 0$$

Find an explicit formula for the extremal.

3. Use a transform method to solve the initial value problem

$$\begin{aligned} u_{tt} &= u_{xx}, \quad x \in R, \quad t > 0 \\ u(x,0) &= f(x), \quad u_t(x,0) = 0, \quad x \in R \end{aligned}$$

Make any assumptions on f that are required.

4. Consider the initial value problem

$$\begin{aligned} u_t + uu_x + au &= 0, \quad x \in R, \quad t > 0; \quad a > 0 \\ u(x,0) &= f(x), \quad x \in R \end{aligned}$$

Determine a nontrivial condition on f that ensures the solution will exist for all $t > 0$.

5. A one-dimensional model for the density of some quantity u is given by

$$u_t + c(u)u_x = (D(u)u_x)_x + \lambda(x,t)u$$

(a) Give a derivation of this model equation, explaining precisely the origin of each term and what physical processes are involved.

(b) What would the model equation be in three dimensions?

(c) Take the functions c, D and λ to be constant, and use a dispersion relation to make some conclusions about the behavior of solutions to the model equation.

6. Consider a nonlinear model

$$u_t = u_{xx} + f(u), \quad 0 < x < \pi, \quad t > 0$$

subject to boundary conditions

$$u_x(0, t) = u_x(\pi, t) = 0, \quad t > 0$$

Suppose $u(x, t) = u_0 = \mathit{constant}$ is an equilibrium solution. Give an argument, which will require formulation of some basic concepts, to show that if $f'(u_0) < 0$, then u_0 is stable to small perturbations. Hint: obtain a linearized equation for the perturbation and either use an energy estimate or find the solution.