Mathematics 842-843; 824-843
PhD Qualifying Examination–June 1999

This is a four hour exam. Write on one side of your paper only. Work exactly six problems.

1. (Estate Planning) Let $y = y(t)$ be one's total capital at time t and let $r = r(t)$ be the rate that he expends his capital. Let $U = U(r)$ denote his rate of enjoyment, depending on his rate of expenditure. Then his total enjoyment over the time interval $0 < t < T$ is

$$E = \int_0^T e^{-\beta t} U(r(t))dt$$

where the decaying expontial factor reflects the notion that future enjoyment is counted less than today's enjoyment. Initally, his capital is S, and he wants it to be zero at time T. His money gains interest α so that y satisfies the equation

$$\frac{dy}{dt} = \alpha y - r(t)$$

Assume $0.5\alpha < \beta < \alpha$. Determine $r(t)$ and $y(t)$ so that his total enjoyment is maximized in the case that the enjoyment function is given by $U(r) = 2\sqrt{r}$.

2. Using perturbation methods, find a uniformly valid approximation to the boundary value problem

$$\begin{aligned} \epsilon u'' + u' &= a + bx, \quad x \in (0,1), \quad 0 < \epsilon \ll 1 \\ u(0) &= 0, \quad u(1) = 1 \end{aligned}$$

3. Transient temperatures in an infinite rod ($x \in R$) satisfy the heat equation

$$u_t = u_{xx}$$

At time $t = 0$ (only) a unit amount of heat energy is injected at $x = 0$ giving rise to a temperature distribution $u = u(x,t)$. Suppose that all temperatures below some small value δ are imperceptible and hence, for all practical purposes, are zero. Sketch the region in space–time where the "effective" temperature is non-zero, i.e., sketch the domain

$$B = \{(x,t) \mid u(x,t) \geq \delta\}$$

4. Consider the integro-differential equation

$$Lu \equiv -u'' + 4\pi^2 \int_0^1 u(y)dy = \lambda u, \qquad u(0) = u(1) = 0$$

Use energy methods to prove that the eigenvalues of L are positive. Then find all the eigenfunctions corresponding to the eigenvalue $\lambda = 4\pi^2$.

5. Consider the Cauchy problem

$$\begin{aligned} u_t + cu_x &= -\frac{u}{k+u}, \quad x \in R, \ \ t > 0; \quad c, k > 0 \\ u(x,0) &= f(x), \quad x \in R \end{aligned}$$

By solving the problem, determine if the solution exists for all time $t > 0$.

6. Let $u = u(x,t)$, where $x \in \Omega \subset R^n$, $t > 0$, be a positive solution to

$$u_t = \Delta u + f(x)$$

and let $s = \ln u$ denote the local entropy. Assume Ω is a nice, bounded region. Prove that

$$\frac{dS}{dt} + \Phi \geq Q$$

where

$$\begin{aligned} S &= \int_\Omega s \, dx \quad \text{(total entropy in } \Omega\text{)} \\ \Phi &= \int_{\partial\Omega} \nabla s \cdot n dA \quad \text{(entropy flux through } \partial\Omega\text{)} \\ Q &= \int_\Omega f u^{-1} dx \quad \text{(entropy change due to sources in } \Omega\text{)} \end{aligned}$$

7. (Pipe problem) Refer to the accompanying figure. A pipe, insulated on the outside, of inner radius r and outer radius R, and having length L, carries water at velocity V from a hot water heater of temperature T_0 to a faucet on the other end. Initially the temperature of the pipe and the water in the pipe is zero degrees. The basic problem is to formulate an initial boundary value problem that governs the evolution of the temperatures $T_w(x,t)$ and $T_p(x,t)$ of the water and of the pipe, respectively, and then obtain an approximate solution.

Discussion. Assume that the heat capacities (calories per volume per degree) of the water and pipe material are C_w and C_p, respectively. Also, assume no heat flux in the pipe in the x-direction, and ignor all diffusion processes. Note that heat energy is lost from the water to the pipe; assume Newton's law of cooling, which states that the heat loss is proportional to the area and to the temperature difference between the water and the pipe. Use h (measured in calories per area per second per degree) as the proportionality constant (called the heat loss coefficient).

Carefully formulate the mathematical model consisting of differential equations, initial conditions, and boundary conditions. Use u and v for the dimensionless temperatures in the water and pipe, respectively. Note that there are two time scales, t_l representing the heat loss rate, and t_c representing convection. Determine these scales. Scale time by the convection time scale t_c and reformulate the problem in dimensionless form. At this point one might try to make an approximation based on the assumption $t_l \ll t_c$. Discuss thoroughly.