June 2002 Applied Math (842-3) Qualifying Exam

Do three of the first four, and two of the last three.

1. Suppose that there is a unit-free law

$$g(P, l, m, t, \rho) = 0,$$

where P is pressure, l length, m mass, t time and ρ density. Show there is an equivalent law of the form

$$G\left(\frac{l^3\rho}{m}, \frac{t^6 P^3}{m^2\rho}\right) = 0.$$

2. Consider the two-point boundary value problem

$$(P_1) \begin{cases} \epsilon y'' + 2y' + y = 0 \quad \text{for } 0 < x < 1, \\ y(0) = 0, \\ y(1) = 1, \end{cases}$$

where $0 < \epsilon << 1$.

a. Find an approximate solution valid uniformly on $[0, 1]$ as $\epsilon \to 0$.

b. In what sense is the approximation valid "uniformly" on $[0, 1]$?

3. Let

$$I(\lambda) = \int_0^1 e^{\lambda(2t-t^2)} \sqrt{1+t}\, dt.$$

Show that to leading order,

$$I(\lambda) \sim e^{\lambda} \left(\frac{\pi}{2\lambda}\right)^{\frac{1}{2}} \quad \text{as} \quad \lambda \to \infty.$$

4. Does the problem

$$\begin{cases} u'' + u' - 2u = f(x) \quad \text{for } 0 < x < 1, \\ u(0) = 0, \\ u'(1) = 0. \end{cases}$$

have a Green's function? If you think so, find it. If you think not, explain why.

5. A substance is confined in a thin tube between $x = 0$ and $x = L$. The density of the substance at the point x at time t is $u(x,t)$. It moves by diffusion only (with diffusion coefficient $D > 0$), and is *depleted* by a chemical reaction at a rate proportional to its density, (with constant of proportionality $c > 0$).

a. Suppose that the ends of the tube are sealed, so that the substance cannot diffuse across the endpoints $x = 0$ and $x = L$, and that the initial density is $u(x,0) = f(x)$. Formulate an initial-boundary value problem for u.

b. Solve the problem from part (a) with $L = 1$, $D = 1$ and $c = 2$.

6. Let

$$G(x,t) = (4\pi D t)^{\frac{-1}{2}}\, e^{-\frac{x^2}{4Dt}},$$

where $D > 0$. Derive the representation

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)g(y)\,dy + \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s)f(y,s)\,dy\,ds,$$

for the solution to the linear diffusion problem

$$\begin{cases} u_t = Du_{xx} + f(x,t), & \text{for } -\infty < x < \infty,\ t > 0, \\ u(x,0) = g(x). \end{cases}$$

Assume what you want about f and g.

7. Consider the initial value problem

$$(P_3)\begin{cases} u_t + u^3 u_x = 0 & \text{for } -\infty < x < \infty,\ t > 0, \\ u(x,0) = \varphi(x), \end{cases}$$

where

$$\varphi(x) = \begin{cases} 1 & \text{for } x < 0, \\ 1 - x^2 & \text{for } 0 \le x \le 1, \\ 0 & \text{for } x > 1. \end{cases}$$

a. Determine the breaking time t_b.

b. Use the Rankine-Hugoniot condition to determine a solution to (P_3) for $t > t_b$.