Applied Math (842-843) Qualifying Exam **June 2003**

Instructions: You are to work three problems from each part below for a total of six problems. If you attempt more than six problems, clearly indicate which you wish to be graded. All problems have equal value. If you have questions about the wording of a particular problem, ask for clarification. In no case should you interpret a problem in such a way that its solution becomes trivial. Time: 3 hours.

Part I

1. Find the smooth extremals for the functional

$$J(y) = \int_0^3 e^{2x}(y'^2 - y^2)dx$$

subject to the conditions that $y(0) = 1$ and $y(3)$ is unspecified.

2. Consider the nonlinear "Fitzhugh-Nagumo" system

$$\begin{aligned} \frac{dx}{dt} &= x(1-x)(x-a) - y \\ \frac{dy}{dt} &= bx - y \end{aligned}$$

where $b > 0$ and $0 < a < 1$.
(a) In the case when there is just a single critical point, find the critical point and classify it as to type and stability. Sketch the phase portrait in this case.
(b) Let the parameter a be fixed. As the parameter b decreases, a bifurcation occurs and a single additional critical point appears at some critical value of b. Find that critical value in terms of a.

3. Find the first two nonzero terms in the asymptotic expansions for $\displaystyle\int_0^{\pi/2} e^{-x\sin^2 t}\,dt$ as $x \to 0$ and $x \to \infty$.

4. Consider the boundary problem

$$\epsilon y'' + e^{-x}y' + y = 0, \quad y(0) = 0, \quad y'(1) = -e, \quad 0 < \epsilon \ll 1.$$

(a) Show that regular perturbation fails for this problem.
(b) Determine the leading-order uniform approximation for y and use it to estimate $y(1)$ to leading order.

Part II

5. For $0 \le x \le 1$ let L be the differential operator defined by $Lu = u'' - 2xu'$ and let f be continuous on $[0,1]$. For the problem

$$\begin{aligned} Lu &= f(x), \quad \text{for } 0 < x < 1, \\ u(0) &= 0, \\ u'(1) &= 0, \end{aligned}$$

either find the Green's function or explain how you can be sure that there isn't one.

6. Solve the integral equation

$$\int_0^1 e^{x+y} u(y)\, dy - \lambda u(x) = f(x),$$

taking into account all cases.

7. Consider the conservation law $u_t + f(u)_x = 0$, $f \in C^2$, $t > 0$, $x \in \mathbb{R}$.
(a) Assume that $u(x,t) = u_0 + \epsilon u_1(x,t) + \mathcal{O}(\epsilon^2)$, $|\epsilon| \ll 1$, u_0 a constant, u_1 and its derivatives are $\mathcal{O}(1)$, and derive a first order approximation to $u(x,t)$. This approximation is a wave. What is its speed?
(b) Assume that

$$u(x,0) = \begin{cases} u_l, \ x < 0 \\ u_r, \ x \geq 0 \end{cases}$$

where $u_l > u_r \geq 0$ are constants and find the solution to this Riemann problem. This solution is also a wave. Show that as u_r approaches u_l, the speed of this wave approaches that of the first order approximation of (a).

8. Consider the Navier-Stokes equations for a homogeneous viscous flow of constant density $\rho = \rho_0$ and kinematic viscosity $\nu = \mu/\rho_0$, namely

$$\bigtriangledown \cdot \mathbf{v} = 0 \text{ and } \mathbf{v}_t + \mathbf{v} \cdot \bigtriangledown \mathbf{v} = -\frac{\bigtriangledown p}{\rho_0} + \nu \bigtriangleup \mathbf{v}.$$

(a) Assume that flow is two dimensional, say $\mathbf{v} = (u(x,y,t), v(x,y,t))$. Write out the three scalar equations that comprise the Navier-Stokes equations in this case.
(b) Use these equations to show that the only pressure function that allows a steady state horizontal flow (this means the component $v \equiv 0$) are pressure functions of the form $p(x) = ax + b$, where a, b are constants.