Applied Mathematics (842-843)
Qualifying Examination
June 2005

Instructions: *Solve every problem, showing your work neatly and carefully. A table of Laplace transforms is included with the examination.*

1. (14 pts) One version of the Ramsey growth model in economics involves minimizing the "total product"

$$J(M) = \int_0^T (aM - M' - b)^2 dt, \quad a, b > 0,$$

over a fixed planning period $[0, T]$, where $M = M(t)$ is the capital at time t and $M(0) = M_0$ is the initial capital. If $M(t)$ minimizes J, find the capital $M(T)$ at the end of the planning period.

2. (14 pts) A model of a predator–prey system where the predator consumes the eggs of the prey is given by

$$\begin{aligned} \frac{dU}{d\tau} &= -mU + ahX, \\ \frac{dX}{d\tau} &= -hX + bU - \frac{cXP}{B+X}, \\ \frac{dP}{d\tau} &= rP\left(1 - \frac{P}{K}\right) + \frac{ycXP}{B+X}, \end{aligned}$$

where U is the prey population, X is the number of eggs, P is the predator population, and τ is time.

(a) Make a table of the nine constants in the problem, showing their dimensions.

(b) Introduce dimensionless quantites u, x, p, and t for the dependent variables and time, and reformulate the equations in dimensionless form, defining any dimensionless constants you introduce.

3. (14 pts) Consider the *Liouville* equation

$$y'' + (\lambda^2 x^2 + x)y = 0, \quad x > 0,$$

where λ is a very large parameter. Because we expect rapidly oscillating solutions (why?), we make the transformation $y(x) = e^{i\lambda u(x)}$, where $u(x)$ is a real-valued function. Apply a perturbation method, as used in the WKB approximation, to find two real, leading order approximations to two independent solutions to the Liouville equation.

4. (14 pts) The modified *Bessel* function $I_n(x)$ has integral representation

$$I_n(x) = \frac{1}{\pi} \int_0^\pi e^{x\cos\theta} \cos n\theta d\theta, \quad n = 1, 2, 3, ...$$

Find the leading order asymptotic approximation of $I_n(x)$ as $x \to +\infty$.

5. (14 pts) A straight, narrow tube extends from $x = 0$ to $x = \infty$. (It is long enough to be considered semi-infinite.) A chemical reaction at $x = 0$, initiated at time $t = 0$, generates a compound A that diffuses outward into the tube with diffusion coefficient $D > 0$. The reaction is controlled so as to make the density of A at $x = 0$ and time $t > 0$ equal to $f(t)$, where f is a smooth, bounded function, and $f(0) = 0$. For $t \geq 0$, the density of A tends to zero as $x \to \infty$.

   (a) Formulate an initial-boundary value problem for the density $u = u(x,t)$ of A.

   (b) Solve the problem from part (a) and write your solution in the form

$$u(x,t) = \int_0^t K(x, t-\tau) f(\tau)\, d\tau,$$

   giving the function K explicitly.

6. (14 pts) Let the function $k(x,y)$, defined on the square $[0,\pi] \times [0,\pi]$, be given by

$$k(x,y) = \begin{cases} \cos y \, \sin x, & x < y, \\ \sin y \, \cos x, & x > y. \end{cases}$$

   Find the eigenvalues and eigenfunctions of the integral operator $\mathcal{K}$ defined by

$$\mathcal{K} f(x) = \int_0^\pi k(x,y) f(y)\, dy.$$

7. (16 pts) Consider the initial-value problem

$$\begin{cases} u_t + u^2 u_x = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases}$$

   where f is given by $f(x) = \begin{cases} 1, & x \leq 0, \\ 1 - x, & 0 < x < 1, \\ 0, & x \geq 1. \end{cases}$.

   (a) Find the characteristics and draw an accurate characteristic diagram.

   (b) Find the breaking time t_b.