Mathematics 842-843 June 2006
Masters/Qualifying Examination

Work all the problems in order. Write on one side of the sheet only. Show your work carefully.

1. (15 points) Find a uniformly valid approximation to the boundary value problem

$$\varepsilon y'' - (2x+1)y' + 2y = 0, \quad x \in (0,1),$$
$$y(0) = 2, \quad y(1) = 0,$$

where ε is a small, positive parameter.

2. (15 points) Find the solution $u = u(x,t)$ to the partial differential equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 1$$

that takes the value x on the straight line $x = \frac{1}{2}t$, and determine the region in the xt plane where it is valid (i.e., a classical solution).

3. (15 points) Find the Green's function associated with the problem

$$-y'' = \delta_\xi, \quad \xi \in (0,1),$$
$$y(0) - 2y'(0) = 0, \quad y(1) + 2y'(1) = 0.$$

4. (15 points)Consider the integral operator defined by $Ku = \int_0^1 (2x - 9y)u(y)dy$.

(a) Find the eigenvalues and eigenfunctions of K.

(b) For what values of a and for what continuous functions f does the nonhomogeneous problem $Ku - au = f$ **not** have a solution?

5. (20 points) Certain insects lay eggs on plant stems. If X denotes the egg population and Y denotes the biomass of the plant, then the dynamics is given by

$$\frac{dX}{dt} = aY - bX,$$
$$\frac{dY}{dt} = ke^{-cX/Y} - dY,$$

where a, b, c, d, and k are positive constants.

(a) Make a table of all the variables and constants and show their dimensions.

(b) In a simple way, nondimensionalize the system, introducing two appropriate dimensionless parameters (call them r and s). Do this in such a way that the exponential term in the biomass equation is $e^{-x/y}$, where x and y are the dimensionless, dependent variables.

(c) Find the positive equilibrium solution(s) and determine the values of r and s for which the solution(s) are asymptotically stable.

(d) Sketch a phase diagram that, in the case of stability, indicates the nullclines, the equilibria, and key orbits.

6. (20 points) Here is the background for this problem—The second-order, linear differential equation

$$Ly \stackrel{\text{def}}{=} -(ry')' = \lambda r y, \quad r > 0,$$

with $\lambda > 0$, has general solution

$$y = y(r) = aJ(\sqrt{\lambda} r) + bY(\sqrt{\lambda} r), \quad a, b \text{ arbitrary constants.}$$

The function $J = J(z)$ is a even, bounded, C^∞ function on $\mathbb{R}$; also, $J(0) = 1$ and J oscillates and decays for $z > 0$, having positive zeros at $z = z_k$, $k = 1, 2, 3, \ldots$ with $\lim_{k\to\infty} z_k = +\infty$. The function $Y = Y(z)$ is C^∞ for $z > 0$ and it has a logarithmic-type singularity at $z = 0$. Finally, the Laplacian in polar coordinates is

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

(a) Find the eigenvalues and eigenfunctions for the problem $Ly = \lambda r y, \quad 0 < r < R, \quad y(0)$ bounded, $y(R) = 0$, and prove that the eigenfunctions are orthogonal with respect to an appropriate weighted inner product.

(b) Consider a metal, laminar, circular plate of radius R and diffusivity k that is insulated on its top and bottom faces so that heat flows only in the xy plane. At time $t = 0$ the plate has a given, radial temperature distribution $f(r)$, $0 \le r \le R$, and the boundary $r = R$ of the plate is held at zero degrees for all time $t > 0$. Formulate an initial, boundary value problem that governs the temperature in the plate at any location and time.

(c) Find a formula for the solution to the problem in part (b). Define carefully any symbols you use, norms or inner products you introduce, and so on.