June 2007 Applied Mathematics Qualifying Exam

Instructions: You are to work three problems from each part below for a total of six problems. If you attempt more than six problems, clearly indicate which you wish to be graded. All problems have equal value. If you have questions about the wording of a particular problem, ask for clarification. In no case should you interpret a problem in such a way that its solution becomes trivial. Time: 3 hours.

Part I

1. Assume that the thermal conductivity K of a gas, mean free path λ, the average molecular velocity v, the mass density ρ and the specific heat at constant volume c satisfy a unit free law

$$F(K, \lambda, v, \rho, c) = 0. \tag{1}$$

Let L be length, T time, M mass and Θ temperature. Given the dimensions

$$[K] = MLT^{-3}\Theta^{-1}, \quad [\lambda] = L, \quad [v] = LT^{-1}, \quad [\rho] = ML^{-3} \quad \text{and} \quad [c] = L^2T^{-2}\Theta^{-1},$$

show that

$$K = A\lambda v\rho c,$$

for some dimensionless constant A.

2. Assume that the solution to the boundary value problem

$$(P_2)\begin{cases} \varepsilon y'' + (1+\varepsilon)y' + y = 0, & \varepsilon \ll 1,\ 0 < x < 1, \\ y(0) = 0, \quad y(1) = 1, \end{cases}$$

has a boundary layer at $x = 0$.

a. Find an approximate solution that is valid in the outer region $x = O(1)$.

b. Use balancing to determine the width $\delta(\varepsilon)$ of the boundary layer. (Note that you have four coefficients from which to extract a two-term dominant balance. You needn't check every pair. Just find the right one.)

c. Rescale the equation in the boundary layer. Find an approximate solution that is valid for $x = O(\delta(\varepsilon))$.

d. Use matching to paste your approximate solutions together in some intermediate region. Give an approximate solution y_u that is valid uniformly on $[0, 1]$ as $\varepsilon \downarrow 0$.

3. Let $q_1, \ldots, q_n$ be generalized coordinates in some configuration space $\mathcal{C}$. Define the functional

$$A(q) = \int_a^b L(t, q(t), \dot{q}(t))\, dt,$$

for smooth curves $q(t) = (q_1(t), \ldots, q_n(t))$ such that $q(a) = \alpha$ and $q(b) = \beta$, where α and β are fixed points in $\mathbf{R}^n$.

a. For extremals q of A, derive the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0, \quad k = 1, \ldots, n.$$

b. Define the generalized momentum $p(t) = (p_1(t), \ldots, p_n(t))$ and the Hamiltonian $H = H(t, q, p)$.

c. Use the results of parts (a) and (b) to derive Hamilton's equations,

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \text{for } j = 1, \ldots, n.$$

4. Show that

$$\int_0^1 e^{t-1} t^x (1+t^2)^{-x}\, dt \sim \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} 2^{-x},$$

to leading order, as $x \to \infty$.

Part II

5. Let Q be a domain in $\mathbf{R}^3$ bounded by the smooth, closed surface ∂Q. Consider the initial-boundary value problem for the linear Klein-Gordon equation on Q:

$$(P_5) \begin{cases} w_t - \Delta w + w = 0, & \text{for } x \in Q,\ t > 0, \\ w(x,0) = f(x), \\ w_t(x.0) = g(x), \\ w\big|_{\partial Q} = 0. \end{cases}$$

a. Let w be a smooth solution to (P_5). Derive the energy identy

$$E(t) = \frac{1}{2} \int_Q \left[w_t(x,t)^2 + |\nabla w(x,t)|^2 + w(x,t)^2 \right] dx = \text{Constant.}$$

b. Use the result of part (a) to establish the unicity of the smooth solution w to (P_5).

6. Consider the initial-boundary value problem for the heat equation,

$$(P_6) \begin{cases} u_t - u_{xx} = g(x,t), & \text{for } 0 < x < 1,\ t > 0, \\ u(x,0) = f(x), \\ u_x(0,t) = 0, \\ u_x(1,t) = 0. \end{cases}$$

Use separation of variables to derive the solution representation

$$u(x,t) = \int_0^1 G(x,y,t) f(y)\, dy + \int_0^t \int_0^1 G(x,y,t-s)\, g(y,s)\, dyds.$$

7. Let D and c be real numbers, with $D > 0$. Solve the initial value problem

$$(P_7) \begin{cases} u_t - D u_{xx} + c u_x = 0, & \text{for } x \in \mathbf{R},\ t > 0, \\ u(x,0) = f(x). \end{cases}$$

8. Consider the initial-value problem for the quasilinear conservation law,

$$(P_8) \begin{cases} u_t + u u_x = 0 & \text{for } x \in \mathbf{R},\ t > 0, \\ u(x,0) = g(x), \end{cases}$$

where

$$g(x) = \begin{cases} 1 & \text{for } x \le 0, \\ \cos x & \text{for } 0 < x \le \pi/2, \\ 0 & \text{for } x > \pi/2. \end{cases}$$

Does $u_x(x,t)$ blow up in finite time? If so, find the breaking time t_b and breaking point x_b.