Applied Mathematics (Math 842-843)
Qualifying Examination June 2008

Instructions: *Work all six problems. (4 hours)*

1. Find a uniform approximation to the nonlinear boundary value problem

$$\begin{aligned} \varepsilon y'' + 2y' + e^y &= 0, \quad 0 < x < 1, \\ y(0) &= y(1) = 0, \end{aligned}$$

where $0 < \varepsilon \ll 1$.

2. Consider the "signalling" problem

$$\begin{aligned} u_t + \frac{x}{x+1} u_x &= 0, \quad x > 1,\ t \in \mathbb{R}, \\ u(1,t) &= h(t),\ t \in \mathbb{R}, \end{aligned}$$

where h is a known, given function. Find the solution.

3. Consider the functional

$$J(y,z) = \int_0^1 F(y', z')dx,$$

where $y, z \in \mathrm{C}^2[0,1]$, and where F is a given twice continuously differentiable function in each variable. Prove that if F satisfies the condition

$$\det \begin{pmatrix} F_{y'y'} & F_{y'z'} \\ F_{z'y'} & F_{z'z'} \end{pmatrix} \neq 0,$$

then the extremals necessarily have the form $y = ax + b$, $z = cx + d$, where a, b, c, and d are constants.

4. In $\mathbb{R}^3$, an invicid, incompressible fluid of constant density is rotating in steady-state in a bucket of finite radius occupying $-\infty < z \leq 0$, $x^2 + y^2 \leq r^2$. The velocity field is given by

$$\mathbf{v} = (u, v, w) = (-ay, ax, 0),$$

where a is the constant angular velocity. Find the surfaces of constant pressure and hence show that the free surface at the top of the fluid takes the form of a paraboloid.

5. Consider the following dimensional, initial boundary value problem for $T = T(\xi, \tau)$ in heat flow:

$$\begin{aligned} T_\tau &= \alpha T_{\xi\xi}, \quad \xi > 0,\ \tau > 0, \\ T(\xi, 0) &= T_0, \quad \xi > 0, \\ -KT_\xi(0,\tau) &= F, \quad \tau \geq 0, \end{aligned}$$

where K, F, T_0, and α are positive constants.

(a) In the context of heat flow in a bar, explain fully the meaning of the model and state what the various quantities and conditions mean.

(b) Picking specific length, time, and temperature scales, nondimensionalize the problem. (Use x, t, and u for the dimensionless length, time, and temperature.)

(c) Although this problem can be solved exactly by Laplace transforms, here we follow a procedure that leads to an approximate solution. Work with the dimensionless model. Assume there is a time-dependent "layer" $0 \leq x \leq h(t)$ where the temperature is changing, but no heat energy passes into the region $x > h(t)$, where the temperature remains constant. (This layer is not meant to be a small boundary layer.) This implies conditions at $x = h(t)$, *which you should find.* Then follow the steps as indicated.

   i. Define the *total heat content*

$$\theta(t) = \int_0^{h(t)} u(x,t)dx,$$

   and show that $\theta'(t) = 1 + h'(t)$.

   ii. Assuming a quadratic temperature profile, $u = a(t) + b(t)x + c(t)x^2$, $0 \leq x \leq h(t)$, determine a, b, and c from the boundary conditions.

   iii. Find a differential equation for the width of the layer $h(t)$ and determine how $h(t)$ changes as t increases.

   iv. Sketch a few time profiles of the approximate solution $u(x,t)$ to show how the temperature is evolving in time.

6. Consider the nonlinear BVP for $u = u(x,y,t)$ on the region $x \in \mathbb{R}$, $0 < y < 1$, $t > 0$.

$$\begin{aligned} u_t &= \frac{1}{R}(u_{xx} + u_{yy}) - u_{xxxx} - u_{xx}u_y, \\ u(x,0,t) &= 0, \quad u(x,1,t) = 1, \quad x \in \mathbb{R},\ t > 0, \end{aligned}$$

where R is a positive constant.

(a) Show that $u = y$ is an equilibrium solution.

(b) Examine the local (linearized) stability of this equilibrium solution to small perturbations $U = U(x,y,t)$. In particular, consider solutions of the linearized perturbation equations of the form

$$U = e^{\sigma t} e^{ikx} \phi(y).$$

Derive a stability condition depending upon R, the wave number k, and the nth modal form of ϕ.