Math 842-843 Qualifying Exam, June 2009

Do 5 of the 6 problems. Each problem is worth 20 points. Time allowed: 4 hours

1. Let S be the square $(0,\pi)\times(0,\pi)$, in the xy plane, and consider the boundary value problem

$$-\nabla^2 u - 2u = f, \quad (x,y)\in S; \qquad u=0, \quad (x,y)\in\partial S, \tag{1}$$

where f is a smooth function of x and y that vanishes on ∂S. The homogeneous equation $\nabla^2 u = 0$ on the square S with the Dirichlet boundary conditions has eigenvalues $\lambda_{m,n}$ and a complete orthonormal set of eigenfunctions $\varphi_{m,n}$ given by

$$\lambda_{m,n} = m^2+n^2, \quad m,n=1,2,3,\ldots, \qquad \varphi_{m,n} = \sqrt{\frac{2}{\pi}}\sin mx\sin ny, \quad m,n=1,2,3,\ldots.$$

(a) Use the information about the eigenvalues and eigenfunctions of $\nabla^2 u = 0$ to solve the problem (1) for the case $f(x,y)=\sin 2x\sin 5y$.

(b) Discuss the existence and uniqueness of solutions for problem (1) with arbitrary f, stating your results in the form of a theorem that gives appropriate conditions on the function f for the different possibilities.

2. Consider the boundary value problem

$$\epsilon y'' + 2xy' - 4x^2 y = 0, \qquad y(-1)=0, \quad y(1)=e, \qquad \epsilon>0. \tag{2}$$

(a) Find the one-parameter family of solutions to the leading-order outer problem as $\epsilon\to 0$.

(b) Show that the problem (2) cannot have a boundary layer at $x=-1$.

(c) Explain why the problem cannot have a boundary layer at $x=1$ either.

(d) Determine the correct boundary layer structure and leading order solutions by the method of matched asymptotic expansions.

3. (a) (10 points) The problem

$$\epsilon y'' + y' = f', \qquad y'(0)=0, \quad y(1)=0,$$

where f is analytic, $f(1)=0$, and $f'(0)\neq 0$, has the solution

$$y(x) = f(x) + \epsilon\int_0^{\epsilon^{-1}} e^{-t} f'(1-\epsilon t)\,dt - \epsilon\int_0^{\epsilon^{-1}x} e^{-t} f'(x-\epsilon t)\,dt. \tag{3}$$

i. Obtain an asymptotic expansion of the solution (3), up to and including the $\mathcal{O}(\epsilon)$ term, that is valid where $x=\mathcal{O}(1)$.

ii. Let ξ be defined by $x=\epsilon\xi$ and let $Y(\xi)=y(\epsilon\xi)$. Obtain an asymptotic expansion for Y, up to and including the $\mathcal{O}(\epsilon)$ term, that is valid where $\xi=\mathcal{O}(1)$.

iii. Use asymptotic matching to show that the two-term approximations from parts (i) and (ii) match up to (and including) $\mathcal{O}(\epsilon)$.

(b) (10 points) Find a two-term asymptotic expansion as $x\to\infty$ for the integral

$$I(x) = \int_0^1 e^{-xt\ln(1+t)}\,dt.$$

4. Consider the nonlinear partial differential equation

$$u_t - u^2 u_x = 0, \tag{4}$$

where $u = u(x,t)$.

(a) If u is a smooth, nonnegative, integrable function on $x \in \mathbb{R}$ for each $t \geq 0$, and if $u \to 0$ as $|x| \to \infty$ for all t, show that $Q(t) = \int_{\mathbb{R}} u(x,t)dx$ is conserved; that is, $Q(t)$ is constant for for all $t > 0$.

(b) Consider the Cauchy problem consisting of the PDE (4) with initial condition

$$u(x,0) = f(x) = \begin{cases} 1, & x \leq 1, \\ \frac{1}{x}, & x > 1, \end{cases} \quad \text{for } x \in \mathbb{R}.$$

Draw a characteristic diagram.

(c) Find the smooth solution (explicit or implicit) for the Cauchy problem that is valid for all $t > 0$ or until a shock forms.

(d) Determine whether or not a shock forms for this Cauchy problem. State explicitly why or why not. If there is a shock, indicate the point in space-time at which it forms.

5. A voltage drop of V_0 is applied to a solid slab of homogeneous material of thickness H. The electric potential $\Phi(Z)$ and temperature $T(Z)$ are governed by the system

$$\frac{\partial}{\partial Z}\left[S(T)\frac{\partial \Phi}{\partial Z}\right] = 0, \qquad \Phi(0,t) = \frac{V_0}{2}, \quad \Phi\left(\frac{H}{2},t\right) = V_0; \tag{5}$$

$$\rho c \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial Z^2} + S(T)\left(\frac{\partial \Phi}{\partial Z}\right)^2, \qquad \frac{\partial T}{\partial Z}(0,t) = 0, \quad -k\frac{\partial T}{\partial Z}\left(\frac{H}{2}\right) = h\left[T\left(\frac{H}{2},t\right) - T_a\right]; \tag{6}$$

where $S(T)$ is the electrical conductivity, ρ is the density, c is the specific heat, k is the thermal conductivity, h is the convective heat transfer parameter, and T_a is the ambient temperature. Notice that we need consider only half of the slab because of the symmetry.

Now consider the steady-state problem with

$$S(T) = S_a + \beta(T - T_a). \tag{7}$$

(a) Determine appropriate scales for Z, Φ, $T - T_a$, and S, and use them to nondimensionalize the model (5)–(7) (with $\partial/\partial t = 0$) using dimensionless quantities z, ϕ, u, and s, respectively. Your dimensionless model should contain two dimensionless parameters,

$$\lambda = \frac{k}{hH}, \qquad \epsilon = \frac{\beta V_0^2}{k}.$$

(b) Obtain two-term regular perturbation approximations for ϕ and u as $\epsilon \to 0^+$ with $\lambda = L\epsilon$, where L is fixed as $\epsilon \to 0$.

6. In an intermediate stage of tumor growth, the nutrients, e.g., oxygen, must diffuse in from adjacent normal tissue; however, diffusion is insufficient to supply a *necrotic core* that exists at the center of the tumor. Assume that the tumor is a spherical shell with unknown inner radius r_1 and known outer radius r_2. Assume that the concentration c_2 of the nutrient in the normal tissue and the concentration $c_1 < c_2$ in the necrotic core are both known. Given spherical symmetry, the Laplacian operator is

$$\nabla^2 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right),$$

and the radial nutrient flux is $-D\frac{dc}{dr}$, where $c(r)$ is the nutrient concentration (mass/volume) at radius r and D is the diffusion constant in the tumor tissue. Assume that nutrients are taken up by the tumor (the portion outside of the necrotic core only) at a rate k (mass/volume-time).

(a) Set up a boundary value problem for the steady-state oxygen concentration in the living part of the tumor. Be sure to indicate precisely the governing equation and appropriate boundary conditions.

(b) Solve the boundary value problem of part (a) to obtain a single algebraic equation that determines the radius r_1 of the necrotic core in terms of known values c_1, c_2, r_2, k, and D. (You need not solve this equation for r_1.)

(c) Find a leading-order approximation for r_1 that is valid as long as $r_1 \ll r_2$.