Applied Math Qualifier, June 2010

Answer the four problems in Section 1, and two of the four in Section 2. Clearly identify which two of the four problems you want graded.

- The parts of a problem might not be of equal weight.
- The parts of a problem are not necessarily related.
- If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Section 1: Answer all four of these problems

1. (17 points) Find and match the leading-order outer and inner approximations as $\epsilon \to 0$ for

$$\epsilon y'' + \left(1 + e^{-x}\right) y' - \left(1 + e^{-x}\right) y = 1, \qquad y'(0) = 1, \quad y(1) = 0. \tag{1}$$

   Either combine these to make a uniform approximation, or explain why this cannot be done.

2. (17 points) Use the method of Laplace transforms to solve the initial boundary value problem

$$\begin{aligned} u_t + u_x &= 0, \quad x, t > 0, \\ u(x, 0) &= 0 \quad x > 0; \qquad u(0, t) = e^{2t}, \quad t > 0. \end{aligned}$$

   You may use the table provided at the back of this exam.

3. (17 points) A model rocket is launched from the surface of the earth. Assuming that the propulsive force is constant over the time of the flight, and neglecting air resistance, the motion is governed by the model

$$\frac{dv}{dt} = a - \frac{R^2 g}{(x + R)^2}, \qquad v(0) = v_0; \tag{2}$$

$$\frac{dx}{dt} = v, \qquad x(0) = 0; \tag{3}$$

   where x is the height above the surface, v is the velocity, R is the radius of the Earth, g is the surface gravitational constant, and $a < g$ is the rocket acceleration component caused by thrust.

   (a) Nondimensionalize the problem using scales of appropriate magnitudes. The resulting model should have two parameters: a small parameter ϵ and a parameter $\alpha < 1$. It might help to think about the simplified model that neglects the thrust and the inverse square law. Analysis of this simplified model indicates the appropriate scales.

   (b) Obtain a 2-term regular perturbation solution for the dimensionless model in the limit $\epsilon \to 0$.

4. (17 points) In the following you do not need to actually find the eigenvalues and eigenfunctions.

   (a) The boundary value problem

$$\begin{aligned}(x^2y')' &= -\lambda y, \quad 1 < x < 2,\\ y(1) &= 0, \quad y'(2) = 0,\end{aligned}$$

   has a complete set of orthonormal eigenfunctions $\varphi_n(x)$ with associated eigenvalues λ_n, $n = 1, 2, 3, ...$ Show that the eigenvalues must be positive.

   (b) Consider the following initial boundary value problem for a nonhomogeneous diffusion process:

$$\begin{aligned}u_t &= (x^2u_x)_x, \quad 1 < x < 2,\ t > 0,\\ u(1,t) &= u_x(2,t) = 0, \quad t > 0,\\ u(x,0) &= \frac{1}{1+x}, \quad 1 < x < 2.\end{aligned}$$

   Find the solution $u = u(x,t)$ of this problem, defining carefully any quantity you introduce.

Section 2: Do two of the problems in this Section. Make it clear which problems should be graded.

5. (16 points) Consider Burgers' equation

$$u_t + uu_x = 2u_{xx}, \quad x, t \in \mathbb{R},$$

Find all solutions of the form $u(x,t) = U(z)$, $z = x - ct$, where

$$\lim_{z\to-\infty} U(z) = 4, \quad \lim_{z\to+\infty} U(z) = 1,$$

where the wave speed c is to be determined. Draw a rough plot of the solution at some fixed time t.

6. (16 points)

   (a) Use the method of characteristics to find the solution to the initial value problem

$$\begin{aligned}u_t + xu_x &= 0, \quad x \in \mathbb{R},\ t > 0,\\ u(x,1) &= \frac{1}{1+x^2}.\end{aligned}$$

   (b) In spherically symmetric tumor growth, the concentration of nutrients $u = u(r,t)$ inside a tumor $(0 < r < R)$ is governed by diffusion and is given by

$$u_t - \frac{1}{r^2}(Dr^2u_r)_r = S,$$

   where S is the constant supply of nutrients to the tumor and D is the diffusion constant. On the boundary the concentration is constant, or $u(R) = u_1$, and at the center of the tumor, $R = 0$, the *flux* of nutrients is zero. Find the steady-state solution for the nutrient concentration.

7. (16 points)

(a) Consider the functional

$$J(y) = \int_0^1 \left\{ \dot{y}e^{2y} + (y-3)^2 \right\} \, dx,$$

Let the domain of J be $\{y \in C^2(0,1) \mid y(0) = y(1)\}$. Prove that $J(y)$ has an absolute minimum.

(b) Consider the functional

$$J(y) = \int_0^1 \left\{ p(x)(y'(x))^2 + q(x)(y(x))^2 \right\} \, dx,$$

where $p, q \in C^2(0,1)$ and $p(x) > 0$ and $q(x) > 0$ for $x \in [0,1]$. Let the domain of J be $\{y \in C^2(0,1) \mid y(0) = a, y(1) = b\}$. Prove that $J(y)$ has an absolute minimum. Hint: Show that the solution to the Euler Equation is also an absolute minimum.