Math 830-831, Qualifying Exam, January, 2002

1. Apply the LaSalle Invariance Theorem to the system

$$\begin{aligned} x' &= 4y^3 - xy^2 \\ y' &= -3x + x^2 y. \end{aligned}$$

Give a very accurate description of the domain of attraction to the origin. (Hint: Take $V(x,y) = \alpha x^2 + \beta y^4$.)

2. Use the variation of constants formula to solve the difference equation IVP

$$x(t+1) = \begin{bmatrix} -1 & 4 \\ -3 & 6 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 3^t \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

3. Show that if $\Phi(t)$ is a fundamental matrix for the n dimensional vector differential equation $x' = A(t)x$, then $\Psi(t) := \Phi(t)C$, where C is a nonsingular $n \times n$-matrix is also a fundamental matrix. Then prove that all fundamental matrices are of this form.

4. Find the equation(s) of the stable subspace for

$$x(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{4} & -2 \end{bmatrix} x(t).$$

5. Find the equilibrium point(s) for the predator-prey model

$$\begin{aligned} N' &= rN(1 - N/K) - aNP, \\ P' &= bNP - cP, \end{aligned}$$

where we assume $r, K, b, c > 0$. If $bK > c$ determine the stability of the equilibrium point in the open first quadrant.

6. Assume μ_0 is a Floquet multiplier for the Floquet system $x' = A(t)x$, where ω is the smallest positive period of the matrix function $A(t)$.

   (a) Show that there is a nontrivial solution $x_0(t)$ satisfying

   $$x_0(t+\omega) = \mu_0 x_0(t), \quad t \in \mathbb{R}.$$

   (b) Show that there is a nontrivial solution of the form

   $$x(t) = p(t)e^{\alpha t},$$

   where p is a continuously differentiable function on $\mathbb{R}$ which is periodic with period ω and α is a constant.

   (c) Show that if $\mu_0 = -1$ there is a nontrivial solution of the Floquet system which is periodic with period 2ω.