Math 830-831, Qualifying Exam, January, 2005

1. Show that each of the following systems has a nontrivial periodic solution:

   (a)
$$x' = x + y - x(x^2 + 2y^2),$$
$$y' = -x + y - y(x^2 + 2y^2).$$

   (b)
$$x' = x - y - x^3,$$
$$y' = x + y - y^3.$$

2. Prove that μ_0 is a Floquet multiplier of the Floquet system $x' = A(t)x$ iff there is a nontrivial solution x satisfying $x(t + \omega) = \mu_0\, x(t)$ for all t.

3. Using the variation of constants formula solve the IVP

$$x' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

4. Find an appropriate Green's function and use it to solve the BVP

$$(e^{-5t}x')' + 6e^{-5t}x = e^{3t},$$
$$x(0) = 0, \quad x(\log(2)) = 0.$$

5. Prove that the calculus of variations problem

$$Q[x] = \int_0^1 \{e^{-4t}[x'(t)]^2 - 4e^{-4t}x^2(t)\}\, dt$$

   subject to

$$x(0) = 1, \quad x(1) = 0$$

   has a proper global minimum and find this minimum value.

6. Assume p, q, and r are continuous on $[a, b)$, $p(t) > 0$ on $[a, b)$ with $\lim_{t\to b-} p(t) = 0$. Prove that eigenfunctions corresponding to distinct eigenvalues of the SLP

$$(p(t)x')' + q(t)x = \lambda r(t)x$$
$$x, x' \text{ are bounded on } [a, b),$$
$$\gamma x(b) + \delta x'(b) = 0, \quad \gamma^2 + \delta^2 > 0$$

   satisfy a certain orthogonality condition.