Math 830-831, Qualifying Exam, January, 2006

1. Find the Floquet multipliers for the Floquet system $x' = \begin{bmatrix} -2+\cos t & 0 \\ \cos t & -2 \end{bmatrix} x$. What conclusion about stability can you draw from the Floquet multipliers that you found?

2. Given $A = \begin{bmatrix} 3 & -5 \\ 5 & -7 \end{bmatrix}, \quad b(t) = \begin{bmatrix} e^{-2t} \\ 0 \end{bmatrix}$.

   (a) Find a fundamental matrix for the linear system $x' = Ax$.

   (b) Use your answer to solve the IVP $x' = Ax + b(t), \quad x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

3. (a) Use a Lyapunov function of the form $V(x,y) = \alpha x^m + \beta y^n$, for appropriate constants α, β, m, and n, to show that the equilibrium (critical) point $(0,0)$ for the system

$$\begin{aligned} x' &= -y \\ y' &= -yx^2 + 4x^3 \end{aligned}$$

   is globally asymptotically stable.

   (b) Determine the nature and the stability of the critical point $(0,0)$ for the system

$$\begin{aligned} x' &= f(x,y) = 2x + 3xy + x^2 - y \\ y' &= g(x,y) = x + 3y + y^2. \end{aligned}$$

4. Show that the system

$$\begin{aligned} x' &= -y + x(r^4 - 3r^2 + 1) \\ y' &= x + y(r^4 - 3r^2 + 1), \end{aligned}$$

   where $r^2 = x^2 + y^2$, has a periodic solution in the region $0 < r < 1$ and in the region $1 < r < 3$.

5. Using an appropriate Green's function (no credit for any other method), solve the BVP

$$\left(\frac{1}{t^2}x'\right)' + \frac{2}{t^4}x = t^3,$$

$$x(1) = 0, \quad x(2) = 1.$$

6. Show that if $I[x] := \int_a^b f(t, x(t), x'(t))dt$ subject to $x(a) = A$, $x(b) = B$ has a local extremum at $x_0(t)$, then $x_0(t)$ satisfies the Euler–Lagrange equation.

7. Assume p, q, and r are continuous on $[a,b]$, with $p(t) > 0$, $r(t) > 0$ on $[a,b]$. Prove that eigenfunctions corresponding to distinct eigenvalues of the SLP

$$(p(t)x')' + q(t)x = -\lambda r(t)x$$

$$\alpha x(a) - \beta x'(a) = 0, \quad \alpha^2 + \beta^2 > 0$$

$$\gamma x(b) + \delta x'(b) = 0, \quad \gamma^2 + \delta^2 > 0$$

   satisfy a certain orthogonality condition. What results can you state regarding the above Sturm–Liouville problem.