Math 830-831, Qualifying Exam, June, 2004

1. Using an appropriate variation of constants formula, solve the IVP

$$x' = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} e^{-t} \\ 2e^{-t} \end{bmatrix}, \quad x(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

2. Apply the LaSalle Invariance theorem with $V(x,y) := x^2 + y^2$ to the system

$$\begin{aligned} x' &= xy^2 - 3y, \\ y' &= -4y + 3x. \end{aligned}$$

3. Assume throughout this question that $x_1(t)$, $x_2(t)$, $\cdots$, $x_n(t)$ are real-valued functions of class $C^{n-1}(I)$ and let $W(t)$ denote the Wronskian function of these n functions for t in the interval I. Prove or disprove each of the following statements:

   (a) If $W(t_0) \neq 0$ for some $t_0 \in I$, then $x_1(t)$, $x_2(t)$, $\cdots$, $x_n(t)$ are linearly independent on I.

   (b) If $W(t_0) = 0$ for some $t_0 \in I$, then $x_1(t)$, $x_2(t)$, $\cdots$, $x_n(t)$ are linearly dependent on I.

4. Prove that eigenfunctions corresponding to distinct eigenvalues of the SLP

$$(p(t)x')' + q(t)x = \lambda r(t)x$$

$$\alpha x(a) - \beta x'(a) = 0, \quad \alpha^2 + \beta^2 > 0$$

$$\gamma x(b) + \delta x'(b) = 0, \quad \gamma^2 + \delta^2 > 0$$

   satisfy a certain orthogonality condition.

5. State and prove Floquet's theorem.

6. Using an appropriate Green's function, solve the BVP

$$x'' + x = \cos(2t)$$

$$x(0) = x(\pi), \quad x'(0) = x'(\pi).$$