Masters and Qualifying Exam, **May, 2006.**

Work any seven and clearly mark which problem you are omiting.

1. (a) Use Putzer's algorithm to find e^{At} given that $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

   (b) Given B is a 2×2 constant matrix such that $e^{Bt} = \begin{bmatrix} (1-t)e^{2t} & te^{2t} \\ -te^{2t} & (1+t)e^{2t} \end{bmatrix}$, solve the initial value problem
$$x' = Bx + \begin{bmatrix} te^{2t} \\ e^{2t} \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

2. Assume $p(t)$, $q(t)$, and $r(t)$ are continuous on the half open interval $[a, b)$, with $p(t) > 0$, $r(t) > 0$ on $[a, b)$. Show that if $\lim_{t \to b-} p(t) = 0$, then eigenfunctions corresponding to distinct eigenvalues for the singular Sturm-Liouville problem
$$Lx = (p(t)x')' + q(t)x = -\lambda r(t)x,$$
$$\alpha x(a) - \beta x'(a) = 0, \quad \alpha^2 + \beta^2 > 0$$
$$x(t),\ x'(t) \quad \text{are bounded on} \quad [a, b),$$
are orthogonal with respect to the weight function $r(t)$. As a completely separate problem find a constant a so that $x_1(t) = t + a$, $x_2(t) = e^t$ are orthogonal with respect to the weight function $r(t) = e^{2t}$ on $[0, 1]$.

3. (a) Find the minimum value of
$$Q[x] = \int_1^2 \{t(x'(t))^2 + \frac{4}{t}x^2(t)\}\, dt$$
subject to $x(1) = 1$, $x(2) = 4$. Verify that this problem does have a global minimum.

   (b) Show that the problem
$$I[x] = \int_0^{2\pi} \left[e^{2t}x^2(t) - 3e^t x(t)x'(t) - (\sin t)(x'(t))^2\right]\, dt$$
subject to $x(0) = 2$, $x(3) = 1$, has no local extremums.

4. (a) Find the Floquet multipliers for the system
$$x' = \begin{bmatrix} -2 & \cos t \\ 0 & -2 + \cos t \end{bmatrix} x.$$
By just looking at the Floquet multipliers that you found, what stability conclusion can you draw.

   (b) In general, prove that if the Floquet system $x' = A(t)x$ with minimum positive period of $A(t)$ is ω and if μ is a Floquet multiplier, then there is a nontrivial solution $x(t)$ satisfying $x(t + \omega) = \mu\, x(t)$ for $-\infty < t < \infty$.

5. By finding an appropriate Green's function solve the boundary value problem

$$(e^{2t}x')' = e^{3t}, \quad x(0) = 0, \quad x(\log(2)) = 0.$$

Using your answer, solve the boundary value problem

$$(e^{2t}x')' = e^{3t}, \quad x(0) = 3, \quad x(\log(2)) = 0.$$

6. Write the van der Pol equation $x'' + \mu(x^2 - 1)x' + x = 0$ as a 2-dimensional system in the standard way. Determine the stability of the trivial solution for all values of the real parameter μ.

7. Apply the LaSalle Invariance Theorem to the system

$$x' = -y$$
$$y' = -yx^2 + 4x^3.$$

(Hint: $V(x, y) = ax^4 + y^2$).

8. Determine if each of the following has a nontrivial periodic solution or not

(a)

$$x' = x + y - x\left(\frac{3}{2}x^2 + y^2\right)$$
$$y' = -x + y - y\left(\frac{3}{2}x^2 + y^2\right)$$

(b)

$$x' = y$$
$$y' = -x - y + x^2 + 2y^2.$$