Masters and Qualifying Exam, **May, 2007.**

1. Use Putzer's algorithm to find e^{At} given that $A = \begin{bmatrix} 1 & -1 \\ 5 & -1 \end{bmatrix}$.

2. Show that the differential equation $(r(t)x')' + q(t)x = 0$, $r(t) > 0$, has a positive solution on an interval I iff the corresponding Riccati equation has a solution on I.

3. Use the eigenpair method to find a fundamental matrix for

$$x' = Ax = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x.$$

   Use your answer to find e^{At}.

4. Show that the boundary value problem

$$(r(t)x')' = 0, \quad \alpha x(a) - \beta x'(a) = 0, \quad \gamma x(b) + \delta x'(b) = 0,$$

   where $\alpha^2 + \beta^2 > 0$, $\gamma^2 + \delta^2 > 0$, has only the trivial solution if and only if

$$d := \alpha\gamma \int_a^b \frac{1}{r(t)} dt + \frac{\beta\gamma}{r(a)} + \frac{\alpha\delta}{r(b)} \neq 0.$$

5. Prove that if $\Phi(t)$ is a fundamental matrix for the n-dimensional system $x' = A(t)x$, then for any $n \times n$ nonsingular constant matrix C, $\Psi(t) = \Phi(t)C$ is also a fundamental matrix. Proof that all fundamental matrices are of this form.

6. Show for the two dimensional system $x' = Ax$, where A is a constant matrix, that if the trace of A is positive and the determinant of A is not zero, then the origin is unstable.

7. Find the Green's function for the boundary value problem (BVP)

$$x'' + x = 0, \quad x(0) = 0 = x(1).$$

   Use your answer to solve the BVP

$$x'' + x = 1, \quad x(0) = 1, \quad x(1) = 0.$$

8. Draw the phase diagram for the system

$$\begin{aligned} x' &= -y + x(x^2+y^2)\sin\left(\frac{\pi}{\sqrt{x^2+y^2}}\right) \\ x' &= x + y(x^2+y^2)\sin\left(\frac{\pi}{\sqrt{x^2+y^2}}\right). \end{aligned}$$

   Is the origin stable? Is the origin asymptotically stable?