**Math 830-831, Qualifying Exam, June, 2010**

Work all six of the following problems:

1. Show that each of the following systems either has a nontrivial periodic solution or does not have a nontrivial periodic solution.

   (a)
$$x' = x - y - x(2x^2 + y^2),$$
$$y' = x + y - y(2x^2 + y^2).$$

   (b)
$$x' = -y,$$
$$y' = x + y - y^2 - x^2.$$

2. (a) Find the Floquet multipliers for the Floquet system
$$\vec{x}\,' = \begin{bmatrix} \cos(2\pi t) & 1 \\ 0 & \cos(2\pi t) + 1 \end{bmatrix} \vec{x}.$$

   (b) What can you say about a Floquet system if 1 is a Floquet multiplier? What can you say about a Floquet system if $-1$ is a Floquet multiplier? Under what conditions on the Floquet multipliers is the trivial solution of the Floquet system asymptotically stable?

3. (a) Find a fundamental matrix for
$$x' = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} x.$$

   Use a variation of constants formula to solve the initial value problem
$$x' = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

4. (a) Find the Cauchy function for $(tx')' = 0, \quad t > 0$.

   (b) Find the Green's function for the boundary value problem
$$(tx')' = 0, \quad x(1) = 0, \quad x(2) = 0.$$

   (c) Use your Green's function to help you solve the boundary value problem
$$(tx')' = t^2, \quad x(1) = 1, \quad x(2) = 2.$$

5. (a) Prove directly that the self-adjoint differential equation $(p(t)x')' + q(t)x(t) = 0$ is disconjugate on $[a, b]$ if and only if it has a positive solution on $[a, b]$.

   (b) Verify that $x'' + t(t - 2)x = 0$ is disconjugate on $[0, 3]$.

6. Assume $p$, $q$, and $r$ are continuous on $[a, b)$, with $p(t) > 0$, $r(t) > 0$ on $[a, b)$, and $\lim_{t \to b-} p(t) = 0$. Prove that eigenfunctions corresponding to distinct eigenvalues of the singular SLP
$$(p(t)x')' + q(t)x = -\lambda r(t)x$$
$$\alpha x(a) - \beta x'(a) = 0, \quad \alpha^2 + \beta^2 > 0$$
$$x, \quad x' \quad \text{are bounded on } [a, b)$$
   satisfy a certain orthogonality condition.