Math 830-831, Qualifying Exam, June, 2011

1. Find the Cauchy function for the differential equation $(\frac{1}{t^6}x')' + \frac{12}{t^8}x = 0$ and use your answer to solve the initial value problem (IVP)

$$(\frac{1}{t^6}x')' + \frac{12}{t^8}x = \frac{2}{t^6}, \quad x(1) = 1, \quad x'(1) = 2.$$

Also find the Green's function for the boundary value problem

$$(\frac{1}{t^6}x')' + \frac{12}{t^8}x = \frac{2}{t^6}, \quad x(1) = 0, \quad x(2) = 0.$$

2. State and prove the variation of constants formula for the vector IVP

$$x' = A(t)x + b(t), \quad x(t_0) = x_0.$$

3. Using Putzer's algorithm and the variation of constants formula solve the IVP

$$x' = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

4. (a) Use the eigenpair method to find two linearly independent solutions of the vector differential equation

$$x' = \begin{bmatrix} -1 & 6 \\ 1 & 4 \end{bmatrix} x.$$

(b) Prove that your solutions in (a) are linearly independent.

(c) Use (b) to find a fundamental matrix.

(d) Use (c) to find e^{At}, where

$$A := \begin{bmatrix} -1 & 6 \\ 1 & 4 \end{bmatrix}.$$

5. Prove that if $\|\cdot\|_1$ is the matrix norm corresponding to the vector traffic norm (l_1 norm), which is defined by

$$\|x\|_1 = \| \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \|_1 = \max_{1\le k\le n}|x_k|,$$

then $\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^n |a_{ij}|$, where A is an $n \times n$ constant matrix.

6. Consider the self-adjoint DE $(p(t)x')' + q(t)x = 0$, where we assume $p(t) > 0$ on $[a, b]$ and p and q are continuous on $[a, b]$. Prove that no nontrivial solution of this DE has an infinite number of zeros in $[a, b]$.