Math 830-831, Qualifying Exam, June, 2012

Work all six questions. Each problem is worth 20 points.

1. Use Putzer's algorithm to help you solve the IVP

$$x' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix} x + \begin{bmatrix} 2e^{-3t} \\ 0 \\ 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

2. Verify that the Green's function for the BVP

$$x'' + y = 0, \quad x(0) = 0, \quad x\left(\frac{\pi}{2}\right) = 0$$

exists. Then derive this Green's function. Finally, use your answer to solve the BVP

$$y'' + y = 4e^{3t}, \quad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 2.$$

3. Prove that the calculus of variations problem

$$I[x] := \int_1^e \{\frac{1}{t}[x'(t)]^2 - \frac{1}{t^3}x^2(t)\}\, dt$$

subject to

$$x(1) = 0, \quad x(e) = 2$$

has a proper global minimum at some $x_0 \in C^2[1, e]$ and then find $x_0(t)$.

4. (a) (15 points) Find the matrix norm of the matrix $A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -3 & 2 \\ 0 & -1 & -5 \end{bmatrix}$ corresponding to the maximum norm $||\cdot||_\infty$ and the traffic norm $||\cdot||_1$, respectively. Also, find the Lozinski measures $\mu_\infty(A)$ and $\mu_1(A)$. What can you say about the stability of the trivial solution (give a reason for your answer).

   (b) (5 points) Also find the Lozinski measure $\mu_2(B)$ of the matrix $B = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$.

5. Assume $p(t)$ and $q(t)$ are continuous on the interval $I = [a, b]$ with $p(t) > 0$ on I. Show that there is no nontrivial solution of the self-adjoint equation $(p(t)x')' + q(t)x = 0$ with infinitely many zeros in $[a, b]$. Is the above result true if $I = (0, 1]$?

6. (a) Assume that $I = [a, b)$, p, q, r are continuous on I, $p(t) > 0$, $r(t) > 0$ on I, and $\lim_{t\to b-} p(t) = 0$. Show that eigenfunctions corresponding to distinct eigenvalues for the singular Sturm-Liouville problem

$$Lx = (p(t)x')' + q(t)x = -\lambda r(t)x$$
$$\alpha x(a) - \beta x'(a) = 0$$
$$x, x' \quad \text{are bounded on } I$$

   where $\alpha^2 + \beta^2 > 0$, are orthogonal with respect to the weight function $r(t)$ on I.

   (b) Assume that $I = [a, b]$, p, q are continuous on I, with $p(t) > 0$ on I. Prove if the BVP

$$Lx = (p(t)x')' + q(t)x = 0, \quad \alpha x(a) - \beta x'(a) = 0, \quad \gamma x(b) + \delta x'(b) = 0$$

   has only the trivial solution, then the BVP

$$Lx = (p(t)x')' + q(t)x = h(t), \quad \alpha x(a) - \beta x'(a) = A, \quad \gamma x(b) + \delta x'(b) = B$$

   has a unique solution.