Math 830-831, Qualifying Exam, June 2022

Part I. Work THREE of the following problems.

1. Let $A = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & -1 & 2 \end{bmatrix}$.

   (a) Find e^{At} and solve $x' = Ax$, with $x(0) = x_0 \in \mathbb{R}^4$.

   (b) Determine the stable subspace E^s, the unstable subspace E^u, and the center subspace E^c (if they exist).

   (c) Solve $x' = Ax + f(t)$, $x(0) = x_0$ with $f(t) = \begin{bmatrix} \sin t \\ 2t \\ 3e^{-t} \\ 0 \end{bmatrix}$ and $x_0 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. You don't need to evaluate the integrals.

2. Find all equilibria of the following system and determine their stability and type (saddle, node, center, or focus):

$$x' = x(3 - 2x - y),$$
$$y' = y(1 - x - \frac{y}{2}).$$

   Then sketch the phase diagram near each equilibrium. Make your sketch of the solution curves as accurate as you can.

3. (a) Find the local stable manifold, unstable manifold, and center manifold of the origin for the following system

$$x' = -x^3 y + x^4,$$
$$y' = -y - x^2 + xy^2.$$

   (b) Find the equation that the flow satisfies on the center manifold.

   (c) Determine the qualitative behavior near the origin. That is, sketch the phase diagram near the origin.

4. Use the Lyapunov function method to determine the stability of the equilibrium of the origin of the system

$$x_1' = x_1 x_2^2 - x_1 - 2x_2,$$
$$x_2' = x_2^3 + 3x_1 - 3x_2.$$

Part II. Work THREE of the following problems.

1. Solve the following initial value problem:

$$(y+z)z_x + yz_y = x - y, \;\; z = t - 6 \text{ on the initial curve } C: \; x = t, \; y = 1, \; t > 1.$$

   Describe carefully the domain of the solutions.

2. Classify the following PDE:

$$u_{xx} + xu_{yy} = 0.$$

   When the equation is elliptic, find its canonical form.

3. Consider a Sturm-Liouville Problem

$$\begin{cases} -(p(x)y')' + q(x)y = \lambda y, & , a < x < b, \\ \alpha_1 y(a) + \alpha_2 y'(a) = 0, \\ \beta_1 y(b) + \beta_2 y'(b) = 0, \end{cases}$$

   where p, p', and q are continuous functions on $[a,b]$ and $p(x) \neq 0$ for any $x \in [a,b]$; α_1 and α_2 are not both 0. β_1 and β_2 are not both 0. Prove the following:

   (a) All the eigenvalues are real.

   (b) The eigenfunctions $y_n(x)$ and $y_m(x)$ corresponding to distinct eigenvalues λ_n and λ_m are orthogonal, i.e., $\int_a^b y_n(x)y_m(x)dx = 0$, if $m \neq n$.

4. Solve the following problem:

$$\begin{cases} u_t - a^2 u_{xx} = 0, \; x > 0, \; t > 0, \\ u(0,t) = 0, \; t > 0, \\ u(x,0) = e^{-x} - 1, \; x \geq 0. \end{cases}$$