Math 830-831, Qualifying Exam, May 2021

Part I. Work THREE of the following problems.

1. Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -6 & 5 \\ 0 & -5 & 4 \end{bmatrix}$.

   (a) Find e^{At} and solve $x' = Ax$, with $x(0) = x_0 \in \mathbb{R}^3$.

   (b) Determine the stable subspace E^s, the unstable subspace E^u, and the center subspace E^c (if they exist).

2. Use variation of parameters formula to solve the following initial value problem

$$\begin{aligned} x' &= 2x + 3y + t, \\ y' &= 2x + y - 5, \\ x(0) &= x_0,\ y(0) = y_0. \end{aligned}$$

3. Find all equilibria of the following system and determine their stability and type (saddle, node, center, or focus):

$$\begin{aligned} x' &= \tfrac{1}{10}x(10 - y), \\ y' &= \tfrac{1}{20}y(20 - 5x - y). \end{aligned}$$

   Then sketch the phase diagram near each equilibrium. Make your sketch of the solution curves as accurate as you can.

4. (a) Find the local stable manifold, unstable manifold, and center manifold of the origin for the following system

$$\begin{aligned} x' &= -x^2 y, \\ y' &= -y + x^2 + xy. \end{aligned}$$

   (b) Find the equation that the flow satisfies on the center manifold.

   (c) Determine the qualitative behavior near the origin. That is, sketch the phase diagram near the origin.

Part II. Work THREE of the following problems.

1. Solve the following initial value problem:

$$yz_x - xz_y = 2xyz,\ z = 2t^2 \text{ on the initial curve } C:\ x = t,\ y = t,\ t > 0.$$

   Describe carefully the domain of the solutions.

2. Consider the following initial value problem

$$\begin{aligned} u_{tt} &= u_{xx}, \ \ x \in \mathbb{R}, \ \ t > 0, \\ u(x,0) &= \phi(x), \ \ x \in \mathbb{R}, \\ u_t(x,0) &= \psi(x), \ \ x \in \mathbb{R}, \end{aligned}$$

   where ϕ and ψ are given smooth functions. Suppose for some $l > 0$ that $\phi(x) = \psi(x) = 0$ for all $x \notin [-l, l]$.

   (a) Show that at any given point $x^0 \in \mathbb{R}$, there are numbers $T = T(x^0) > 0$ and U such that $u(x^0, t) = U$ for all $t > T(x^0)$. Find $T(x^0)$ and U.

   (b) Let

$$K(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx,\ P(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx.$$

   Show that $K(t) + P(t)$ =constant for $t \geq 0$.

3. Classify the following PDE:

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0.$$

   Find the canonical form of the above PDE and then solve the PDE.

4. Use the method of separation of variables to find an infinite-series representation of the solution of the following initial boundary value problem:

$$\begin{aligned} &u_t - u_{xx} = 0,\ 0 < x < 2\pi,\ t > 0, \\ &u_x(0,t) = 0,\ u(2\pi, t) = 0,\ t > 0, \\ &u(x,0) = 1 - \cos x,\ 0 \leq x \leq 2\pi. \end{aligned}$$

   You may leave integrals in the solution.