Math 830-831, Part I, Qualifying Exam, January, 2016

Work each of the following five problems:

1. Use Putzer's algorithm to help you solve the vector DE

$$x' = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} x.$$

2. Use a variation of constants formula to solve the IVP

$$x' = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 2e^{-2t} \\ 3t \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

3. Write the differential equation

$$x'' + \cos x = 0$$

as a two dimensional system in the standard way and then graph the phase plane diagram for this system. Find an equation of the separatrix for this system.

4. (a) Prove that if μ_0 is a Floquet multiplier of a Floquet system, then the Floquet system has a nontrivial solution $x(t)$ satisfying

$$x(t + \omega) = \mu_0 x(t), \quad t \in \mathbb{R},$$

where ω is the prime period of the coefficient matrix in the Floquet system.

(b) Prove that if $\mu_0 = -1$ is a Floquet multiplier, then the Floquet system has a nontrivial solution which is periodic with period 2ω.

5. (a) Use the eigenpair method to find two linearly independent solutions of the vector DE

$$x' = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} x.$$

(b) Use your answer in (a) to find a fundamental matrix of

$$x' = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} x.$$

(c) **Use part (b)** to find e^{At}, where

$$A = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}.$$

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Work FOUR of the following problems.

1. Find the integral surface of the vector field $V = (xz, yz, -xy)$ containing the given curve C: $x = t, y = -t, z = t + 2,\ t > 0$.

2. Solve the transport equation

$$\begin{aligned} &5zz_x + z_y = 0,\ x \in R,\ y > 0, \\ &z(x, 0) = 2x + 3,\ x \in R. \end{aligned}$$

   Determine where the shock develops. Draw the characteristic curves.

3. Consider the equation $u_{xx} - 4u_{xy} + 4u_{yy} - 8x^2y = 0$.

   (a) Is the equation hyperbolic, parabolic, or elliptic?

   (b) Introduce suitable coordinates to transform the equation into its canonical form.

   (c) Solve the equation by first solving the canonical form.

4. Use the first Green's identity to prove that there is a unique solution to the Dirichlet problem

$$\begin{aligned} &\Delta u = f,\ \text{ in } \Omega, \\ &u = g,\ \text{ on } \partial\Omega. \end{aligned}$$

   Note: The first Green's identity: $\int_\Omega f \nabla^2 g\, dV = \int_{\partial\Omega} f \frac{\partial g}{\partial n}\, d\sigma - \int_\Omega \nabla f \cdot \nabla g\, dV$.

5. Use the separation of variables method to find an infinite-series representation of the solution to the following problem

$$\begin{aligned} &u_{xx} + u_{yy} = 0,\ 0 < x < 2\pi,\ 0 < y < 1, \\ &u(0, y) = 0,\ u(2\pi, y) = 0,\ 0 < y < 1, \\ &u(x, 0) = \sin x,\ u(x, 1) = 1 - \cos x,\ 0 < x < 2\pi. \end{aligned}$$