Math 830-831, Qualifying Exam, May 2023

Part I. Work THREE of the following problems. (Do not interpret the problems in ways to make them trivial. Must show sufficient supporting work for credit.)

1. For the system of linear equations $x' = Ax$ with

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

find the fundamental solution $U(t)$ with $U(0) = I$.

2. Consider the system of equations $x' = C(t)x$ with $C(t) = A + B(t)$ for $x \in \mathbb{R}^n$. Assume the eigenvalues of A lie in the left half of the complex plane excluding the imaginary axis and $\lim_{t\to\infty} B(t) = 0$. Show that every solution $x(t)$ converges to 0 as $t \to \infty$.

3. Use Lyapunov's method to study the stability of the zero solution for the planar system of equations:

$$\begin{cases} x' = y^2 - x^3 \\ y' = -y - 2xy \end{cases}$$

4. Consider the system of equations

$$\begin{cases} x' = -2x \quad + y^2 \\ y' = x + \mu y + xy \end{cases}$$

at the equilibrium point $\mu = 0, (x, y) = (0, 0)$. Find an approximation of a center-manifold of the equilibrium to quadratic terms and identify the type of bifurcation. (Must show derivation work for credit.)

Part II. Work THREE of the following problems. (Do not interpret the problems in ways to make them trivial. Must show sufficient supporting work for credit.)

Notation. Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) = [-\pi, \pi]$ be the 2π-periodic torus. Denote $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $u_t = \frac{\partial u}{\partial t}$, and so on.

1. Let f, $u^0 \in L^2(\mathbb{T})$ be time-independent, mean-zero functions, and let $u = u(t,x)$ be the unique solution to the following problem for the viscous Burgers equation (with $\nu > 0$),

$$\begin{cases} u_t + uu_x = \nu u_{xx} + f, & \text{on } (0,\infty) \times \mathbb{T}, \\ u(0,x) = u^0(x), & \text{on } \mathbb{T}. \end{cases}$$

   Show that the energy $\|u\|^2_{L^2(\mathbb{T})}$ is uniformly bounded in time (that is, u is in $L^\infty(0,\infty; L^2(\mathbb{T}))$, not just $L^\infty(0,T; L^2(\mathbb{T}))$ for all $T > 0$).

2. Let $u^0 \in L^2(\mathbb{T})$. Denote $i = \sqrt{-1}$.

   (a) Find a Fourier series representation for the solution to the following problem

$$\begin{cases} u_t + iu_{xxx} = 0, & \text{on } (0,\infty) \times \mathbb{T}, \\ u(0,x) = u^0(x), & \text{on } \mathbb{T}. \end{cases}$$

   (b) Based on your answer in part (a), describe qualitatively what will happen to the solution for $t > 0$.

3. Assume $g \in C^1(\mathbb{R})$. Solve the following problem for $\rho = \rho(t,x)$ posed on the half-space (i.e., $x \in \mathbb{R}$ with $x > 0$) and for times $t \geq 0$:

$$\begin{cases} \rho_t + 2x\rho_x = \ln(x), \\ \rho(0,x) = g(x). \end{cases}$$

4. Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose u is harmonic in Ω (i.e., $\triangle u = 0$ in Ω). Fix $\vec{x}_0 \in \Omega$ and let α be a multi-index of order $|\alpha| = k$ with corresponding partial derivative $\partial^\alpha u$. Show that for each ball $B(\vec{x}_0, r) \subset \Omega$ there exists a constant C_k such that

$$|\partial^\alpha u(\vec{x}_0)| \leq \frac{C_k}{r^{n+k}} \int_{B(\vec{x}_0,r)} |u(\vec{x})|\, d\vec{x}.$$

   [Hint: Use induction on k, and properties of harmonic functions.]