Math 830-831. Qualifying exam.
May 27, 2016.

- All problems have equal weight, but their parts, e.g., (a), (b), may be valued differently.
- Write on one side of the paper and start each new problem on a new page. Hand your work in order.

Part 1

Solve all 4 out of 5 problems:

1. Diagonalize A and find e^{tA}. Then accurately sketch the phase portrait of the system $\dot{\mathbf{v}} = A\mathbf{v}$ where

$$A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$$

2. Find the stable, unstable, and the center subspaces of $\dot{\mathbf{v}} = A\mathbf{v}$ for each matrix $A : \mathbb{R}^n \to \mathbb{R}^n$. Then state what you can about $\limsup_{t\to\infty} |\mathbf{v}(t)|$ (e.g., finite positive, infinite, or zero) where $\dot{\mathbf{v}} = A\mathbf{v}$ and $\mathbf{v}(0) = \mathbf{e}_j$ for each standard basis vector $\mathbf{e}_j$, $j = 1, 2, \ldots, n$.

   (a) $A = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

3. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous, then solutions of the initial value problem

$$\dot{\mathbf{v}} = f(\mathbf{v}), \quad \mathbf{v}(0) = \mathbf{v}_0$$

   depend continuously on the initial data as a mapping $\mathbf{v}_0 \mapsto \mathbf{v} : \mathbb{R}^n \to C([0,T]; \mathbb{R}^n)$ for any fixed $T > 0$.

4. Use the variation of constants formula to solve

$$\begin{cases} \dot{x} - x &= e^t \\ \dot{y} + y &= t \end{cases}$$

$$x(0) = 1 \quad \text{and} \quad y(0) = \frac{1}{2}\,.$$

5. Find, with justification, a linear system $\mathbf{v}' = A\mathbf{v}$ such that A is diagonal and the corresponding phase portrait is linearly equivalent to the phase portrait of the linearization of

$$\begin{cases} \dot{x} &= yx^2 + 2y - 3x^2 - 6 \\ \dot{y} &= xy - y + 7x - 7 \\ \dot{z} &= zx^2 + y^2 z^3 \end{cases}$$

   at its equilibrium. *Hint: everything factors nicely.*

Turn to the next page for Part 2.

Part 2

Solve 4 out of 5 problems:

1. Find an integral surface of the vector field $\mathbf{V} = (z^2, z^2 y, x^2)$ that contains the curve
$$t \mapsto (\sqrt[3]{t}, 1, 1)$$

2. Let $P(u) := u_t + x u_x$. Consider the differential equation
$$Pu = -u$$
   (a) Find and sketch the characteristic curves of P.
   (b) Bring the equation to the canonical form by using a suitable coordinate change, and then find the general solution to it.

3. Use the Cauchy-Kovalewsky theorem to find the terms of order ≤ 2 of the series solution near $\mathbf{0} = (0,0,0)$ to the Cauchy problem
$$u_t + u_x u_y + u_{yy} = 0$$
$$u(t=0, x, y) = x + y^2.$$

4. Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Suppose $f \in C(\overline{\Omega})$ and $g \in C(\partial\Omega)$. Use the integration by parts formulas to verify that any $C^2(\overline{\Omega})$ solution of the following boundary value problem is unique:
$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} + u = g & \text{on} \quad \partial\Omega \end{cases}$$
($\mathbf{n}$ denotes the exterior unit normal field on the boundary $\partial\Omega$).

5. Find the Fourier series solution to the initial boundary-value problem for the wave equation:
$$u_{tt} - u_{xx} = 0 \quad \text{for} \quad x \in [0, \pi], \quad t \in \mathbb{R}$$
$$u(t, 0) = 0, \quad u(t, \pi) = 0 \quad \text{for} \quad t \in \mathbb{R}$$
$$u(0, x) = \sin(x) \quad \text{and} \quad u_t(0, x) = \sin(2x), \quad x \in [0, \pi]\,.$$
Simplify your answer as much as possible.
Guessing the answer without a proper justification will receive no credit.