Math 830–842 Qualifying exam.
January 18, 2017.

- All problems have equal weight, but their parts, e.g., (a), (b), may be valued differently.
- Write on one side of the paper and start each new problem on a new page. Hand your work in order.

Part I: Math 830

In this part solve 4 out of 5 problems:

1. Consider the following ODE system:
$$\begin{cases} \dot{x} + 2x &= xy \\ 2\dot{y} + y &= x^2 \end{cases}$$
   (a) Find all equilibrium solutions to this system.
   (b) Find the linearization of this system near the zero equilibrium.
   (c) Sketch as accurately as possible the phase portrait for the linearization.

2. Let $A = (a_{ij})$ be an $n \times n$ matrix and $M = \max_{1 \le i,j \le n} |a_{ij}|$.
   (a) Find, with proof, an upper bound in terms M and/or n on the norm of A as a linear operator $\mathbb{R}^n \to \mathbb{R}^n$.
   (b) Now find, with proof, an upper bound on M in terms of $\|A\|$ and/or n.

3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz. For $\mathbf{y}_0 \in \mathbb{R}^n$ let $\mathbf{y}$ denote a solution to the initial value problem (*you do not need to prove that it exists*):
$$\dot{\mathbf{y}} = f(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{y}_0.$$
   (a) Prove that if such a solution is defined on $[0, T]$, then it is unique on that interval.
   (b) Prove that the right-maximal interval of existence is $[0, \infty)$.

4. For each case, find a suitable dimension n and write down an $n \times n$ matrix A such that the dynamical system generated by $\dot{\mathbf{y}} = A\mathbf{y}$ has the following properties:
   (a) Stable subspace of dimension 2, unstable subspace of dimension 1 and center subspace of dimension 1.
   (b) Trivial ($\{\mathbf{0}\}$) unstable subspace, **only real** eigenvalues, and there exists $\mathbf{y}_0 \in \mathbb{R}^n$ such that the solution $\mathbf{y}(t)$ for initial data $\mathbf{y}(0) = \mathbf{y}_0$ is unbounded on $t \in (0, \infty)$.
   (c) Trivial ($\{\mathbf{0}\}$) unstable subspace, **no** real eigenvalues, and there exists $\mathbf{y}_0 \in \mathbb{R}^n$ such that the solution $\mathbf{y}(t)$ for initial data $\mathbf{y}(0) = \mathbf{y}_0$ is unbounded on $t \in (0, \infty)$.

5. Show how to use Putzer's algorithm to compute $\exp(tA)$ with $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ for $a, b \in \mathbb{R}$.

Turn to the next page for Part 2.

Part II: Math 842

In this part solve all 4 problems:

1. Find a two-term asymptotic approximation for

$$f(x) = \int_0^\pi t e^{x \cos t}\, dt$$

as $x \to \infty$.

2. Consider the singular perturbation problem

$$x' = \frac{1}{2}y - x + xy, \qquad x(0) = 1,$$

$$\epsilon y' = -y + x - xy, \qquad y(0) = 0.$$

(a) Determine the leading order solutions in the inner region.

(b) Determine the differential equations for the $O(\epsilon)$ terms in the inner region. **Do Not Solve These Equations!**

(c) Determine the leading order solution in the outer region up to an unknown constant. Note that the solution is not fully explicit.

(d) Use leading order matching to obtain the constant in the outer approximation and verify that the singular perturbation structure is fully consistent.

3. Suppose the outer and inner approximations of a function $y(x)$ are

$$y^O \sim e^{-3x} + \frac{\epsilon}{x} + o(\epsilon), \qquad y^I \sim \left(\sqrt{1 + A_0\xi^{-1}} + e^{-\xi}\right) + \epsilon A_1 \xi + o(\epsilon),$$

where $x = \epsilon\xi$. Use Van Dyke's matching principle to determine A_0 and A_1.

4. The transport equation in a fluid is

$$\frac{\partial C}{\partial T} + \mathbf{U} \cdot \nabla C = D\, \nabla^2 C,$$

where C is the concentration of a solute, $\mathbf{U}$ is the velocity vector, and D is the diffusion coefficient. Assume that the flow takes place in a tube of length L and radius A, and that the velocity is

$$\mathbf{U} = 2U_0 \left(1 - \frac{R^2}{A^2}\right) \mathbf{i},$$

where R is the radial coordinate. Assume no flow conditions at $R = A$ and $C = C_0$ at $X = 0$.

Scale the model, assuming that $A \ll L$ and that the diffusion term is relatively unimportant. For dimensionless parameters, use the Peclet number, defined as $\text{Pe} = U_0 L/D \gg 1$, and the aspect ratio $\beta = A/L \ll 1$. Be sure to split out the vector terms into separate longitudinal and radial components.