# Math 830-842. Qualifying exam.
# May 26, 2017.

- All problems have equal weight, but their parts, e.g., (a), (b), may be valued differently.
- Write on one side of the paper and start each new problem on a new page. Hand your work in order.

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## Part I: Math 830

**Submit work for 4 of Problems 1–5:**

1. **Use Putzer's Algorithm** to compute $e^{\mathbf{A}t}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}.$$

2. Let $\mathbf{A} \in \mathscr{C}([0,\infty);\mathbb{R}^{n\times n})$ and $\boldsymbol{f} \in \mathscr{C}([0,\infty)\times\mathbb{R}^n;\mathbb{R}^n)$ be given, and suppose that $\mathbf{X} \in \mathscr{C}^1([0,\infty);\mathbb{R}^{n\times n})$ is a fundamental matrix solution for the system $\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x}$

   (a) With $t_0 \in [0,\infty)$, produce the **Variation of Parameters Formula** for a solution to the initial value problem

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \mathbf{A}(t)\boldsymbol{x}(t) + \boldsymbol{f}(t,\boldsymbol{x}(t)), \quad t \in [0,\infty), \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \mathbb{R}^n. \end{cases} \tag{IVP}$$

   (b) Assume both of the following:

   - There is a $K < \infty$ such that

$$\|\mathbf{X}(t)\|_{\mathbb{R}^{n\times n}} \le K \quad \text{and} \quad \|\mathbf{X}(t)^{-1}\|_{\mathbb{R}^{n\times n}} \le K, \quad \text{for all } 0 \le t < \infty.$$

   - There is an $M < \infty$ such that

$$\|\boldsymbol{f}(t,\boldsymbol{x})\|_{\mathbb{R}^n} \le M e^{-t}\|\boldsymbol{x}\|_{\mathbb{R}^n}, \quad \text{for all } 0 \le t < \infty, \text{ and all } \boldsymbol{x} \in \mathbb{R}^n.$$

   Prove that there is an $L < \infty$, independent of $\boldsymbol{x}_0 \in \mathbb{R}^n$, such that the solution to the initial value problem (IVP) in part (a) satisfies

$$\|\boldsymbol{x}(t)\|_{\mathbb{R}^n} \le L\|\boldsymbol{x}_0\|_{\mathbb{R}^n}, \quad \text{for all } 0 \le t < \infty, \text{ and all } \boldsymbol{x}_0 \in \mathbb{R}^n.$$

3. Consider the system $\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}$, with

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & -2 \end{pmatrix}.$$

   (a) Identify the stable, center, and unstable manifolds (the stable, center, and unstable subspaces) in $\mathbb{R}^3$, passing through $\mathbf{0}$ for this system.

   (b) Find all $\boldsymbol{x}_0 \in \mathbb{R}^3$ such that any solution satisfying $\boldsymbol{x}(0) = \boldsymbol{x}_0$ remains bounded as $t \to +\infty$.

   (c) Let $\boldsymbol{x}_0$ in the unstable manifold be given, and let $\boldsymbol{x} \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^3)$ be the solution to the system satisfying $\boldsymbol{x}(0) = \boldsymbol{x}_0$. For each $\alpha \in \mathbb{R}$, determine

$$\lim_{t\to\infty} e^{\alpha t}\boldsymbol{x}(t),$$

   if it exists. If it does not exist or whether it exists cannot be determined, state so. Justify all your claims.

4. Consider the following initial value problem

$$\begin{cases} \dot{x}(t) = \big(\cos(\pi t) + k\big)x(t), & t \in [0, \infty), \\ x(0) = 1. \end{cases} \tag{IVP}$$

(a) Find the solution for this initial value problem (IVP).

(b) Identify a monodromy matrix (value) and the Floquet multipliers for this ODE. For which values of $k$ is the 0 solution stable? For which values of $k$ is 0 asymptotically stable? For which values is it unstable?

(c) Let $k \in \mathbb{R}$ be given. Identify the periodic and the exponential components of the Floquet normal form for the solution on $[0, \infty)$ to the initial value problem (IVP).

5. With $\alpha \neq 0$, consider the autonomous system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \alpha x + x^2 - 2xy \\ xy - \alpha y \end{pmatrix}, \quad \text{for } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

(a) For each $\alpha \neq 0$, identify the equilibrium points, and provide the linearization of the system at each equilibrium point.

(b) For each $\alpha \neq 0$, use the linearized system to determine the stability of the system at those equilibrium points where it is possible.

***Turn to the next page for Part 2.***

## Part 2: Math 842

**Math 842 Component of 830/842 Qualifying Exam 05/26/17**

**Submit work for Problems 1–4:**

1. Find a two-term asymptotic approximation for

$$f(t) = \int_0^\infty u e^{-tu^2} J_1(u)\, du, \quad t \to \infty,$$

where $J_1$ is the Bessel function of the first kind of order 1, defined by

$$J_1(x) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{\left(-\frac{x^2}{4}\right)^n}{n!(n+1)!}.$$

2. (a) Build a mathematical model for an SIR disease using the following state variables and processes:

   i. The total population $N$ is comprised of subpopulations of susceptibles $S$, infectives $I$, and recovereds $R$.

   ii. Each of the subgroups has a natural death rate that is proportional to that group's population, with proportionality constant $\mu$.

   iii. The birth rate has two components: a population turnover component $\mu N$, which provides enough births to counter natural deaths, and a population demographics component $rN(1 - N/K)$, which helps counter any additional mortality caused by the disease.

   iv. Susceptibles become infected at a rate proportional to the size of the susceptible population and also to the size of the infective population, with proportionality constant $\beta$.

   v. Individuals leave the infective class at a rate proportional to the population of that class, with rate constant $\gamma$. A fraction $m$ of these infectives die, while the remainder enter the recovered class.

   Note that this model needs three differential equations and an algebraic equation. Use $T$ rather than $t$ for time. Use the algebraic equation to eliminate $R$ from the system, yielding a model with three differential equations.

   (b) The basic reproductive number is the expected number of secondary infections caused by one infective in an otherwise susceptible population over the duration that individual spends in the infective class. Calculate this quantity from first principles by identifying the expected amount of time spent in the infective class and the infection rate per infective when the population (other than the one infective) consists only of susceptibles. This parameter (call it $b$) will be one of the (three) dimensionless parameters in the scaled model. Typically $1 < b < 10$.

   (c) Identify possible time scales in the problem. You should be able to find at least three choices. Identify what each represents in the model and whether it is a short or long scale.

   (d) Scale the model using one of the long time scales. You will need to identify a small parameter $\epsilon$ and an additional $O(1)$ parameter. Your reduced problem ($\epsilon \to 0$) should adequately describe the long-term dynamics of the model, given the assumption that $b$ is also $O(1)$.

3. Find and match 2-term outer and inner asymptotic approximations for the boundary value problem

$$\epsilon F'(x)y'' + y' = 2qF'(x), \qquad y(0) = 0, \quad y(L) = b,$$

where $L > 0$, $b$, and $q$ are constants and $F$ is a smooth function that satisfies the conditions

$$F(0) \neq 0, \qquad F' > 0, \qquad F'(0) = 1, \qquad F(L) = \frac{b}{2q}.$$

For convenience, define parameters

$$\alpha = F(0), \qquad \beta = F'(L), \qquad \gamma = F''(0).$$

Solution notes for the differential equations:

- The differential equation for $y_1$ can be solved exactly even for this abstract problem. It might help to recall the mathematics of energy arguments.
- In case you do not remember all the rules for undetermined coefficients, the following examples may be helpful:

$$y' + 3y = 2xe^{-3y}$$

has a particular solution of the form

$$y_p = (Ax + Bx^2)e^{-3x}$$

and a particular solution for any equation Ly=q, where $L$ is a linear differential operator, can be found by neglecting all but the lowest order term in $L$.

4. The equations of one-dimensional fluid mechanics with constant temperature are conservation of mass,

$$\rho_T + (\rho U)_X = 0,$$

conservation of momentum,

$$\rho U_T + \rho U U_X + P_X = 0,$$

and a barotropic equation of state,

$$P = F(\rho),$$

where we are using capital or Greek letters for dimensional quantities and lower case Latin letters for dimensionless counterparts. The standard state for the system is

$$\rho = \rho_0, \quad P = P_0, \quad U = 0,$$

and the sound speed $c$ is defined by

$$c^2 = F'(\rho_0).$$

The acoustic approximation is that changes in density are small in comparison with $\rho_0$, which we can build into the system by defining a scaled density perturbation $s$ by

$$\rho = \rho_0(1 + \epsilon s),$$

where $\epsilon \ll 1$ is assumed to be known. Use this setting to derive the wave equation as the leading order approximation for $s$. This will require you to scale length by an arbitrary length scale $L$, scale time using the time required for a signal to travel a distance $L$, and scale velocity by a scale $u_r$ that needs to be determined by a dominant balance argument. Note that you do not need to choose a scale for the pressure because you can expand $F(\rho)$ about $\rho_0$.