Math 830-831. Qualifying exam.
January 17, 2018.

- All problems have equal weight, but their parts, e.g., (a), (b), may be valued differently.
- Write on one side of the paper and start each new problem on a new page. Hand your work in order.

Math 830 Component of 830/831Qualifying Exam

Part I: Math 830

Submit work for 3 of Problems 1–4:

1. **Use Putzer's Algorithm** to compute $e^{\boldsymbol{A}t}$, where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

2. Consider the system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$, with

$$\boldsymbol{A} = \begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 2 \\ -1 & -1 & 2 \end{pmatrix},$$

which has the characteristic polynomial

$$P_{\boldsymbol{A}}(\lambda) = -2\lambda^2 - \lambda^3.$$

   (a) Identify the stable, center, and unstable manifolds passing through $\mathbf{0}$ for this system.

   (b) Find all $\boldsymbol{x}_0 \in \mathbb{R}^3$ such that any solution satisfying $\boldsymbol{x}(0) = \boldsymbol{x}_0$ remains bounded as $t \to +\infty$.

3. Fix $k \in \mathbb{R}$, and consider the system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t)$, with $\boldsymbol{A} \in \mathscr{C}^\infty(\mathbb{R}; \mathbb{R}^{2\times 2})$ given by

$$\boldsymbol{A}(t) = \begin{pmatrix} k & 0 \\ \cos(\pi t) & k \end{pmatrix}.$$

   (a) Find the principle matrix solution $\boldsymbol{X} : \mathbb{R} \to \mathbb{R}^{2\times 2}$ at $t = 0$ for this system.

   (b) Identify a monodromy matrix and the Floquet multipliers for the system. Based on the Floquet multipliers, is the $\mathbf{0}$ solution stable, aysmptotically stable, or unstable?

4. Consider the following autonomous ODE system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x\left(1 - x^2 - y^2\right)^2\left(4 - x^2 - y^2\right) - 2y \\ y\left(1 - x^2 - y^2\right)^2\left(4 - x^2 - y^2\right) + 2x \end{pmatrix}, \quad \text{for } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

   (a) Reexpress the system using polar coordinates, and ketch a phase diagram for this system. In your sketch, identify all equilibrium points and separatrices.

   (b) Identify all ω-limit sets, ω-limit cycles, α-limit sets, and α-limit cycles.

   (c) For each ω-limit set identified in part (b), describe the associated basin of attraction.

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Math 831 Component of 830/831Qualifying Exam

Part 2: Math 831

Submit work for 3 of Problems 1–4:

1. Find all integral surfaces of the vector field $\mathbf{V} = (x, y, xy(z+1))$ containing the given curve C: $x = t, y = 2t^2, z = t+1,\ t > 0$.

2. Verify that the IVP

$$u_t = 2uu_x + u_x^2,$$
$$u(0,x) = 1 + 2x - 3x^2.$$

   satisfies the requirements for Cauchy-Kovalevsky Theorem about the origin, and explicitly find all terms of degree three and less.

3. Classify the PDE

$$2u_{xx} - 8u_{xy} + 8u_{yy} + 3u + xe^y = 0.$$

   Then solve this PDE by first solving a corresponding canonical form.

4. Use the separation of variables method to find an infinite-series representation of the solution to the following problem

$$\begin{cases} u_t = u_{xx},\ 0 < x < \pi,\ t > 0, \\ u(0,t) = 0,\ u_x(\pi,t) = 0, t > 0, \\ u(x,0) = \frac{1}{3}\sin(4x),\ 0 < x < \pi. \end{cases}$$