Math 830-842. Qualifying exam.
January 23, 2019.

- All problems have equal weight, but their parts, e.g., (a), (b), may be valued differently.
- The parts of a problem are not necessarily related.
- If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.
- Write on one side of the paper and start each new problem on a new page. Hand your work in order.

Math 830 Component of 830/842Qualifying Exam

Submit work for three of the four problems below.

1. (a) State the Picard-Lindelöf Theorem.

   (b) Find the general term for the Picard approximations of the system

$$\begin{cases} x'(t) = 2t(1 + x(t)), \quad t > 0 \\ x(0) = 0. \end{cases}$$

   Does the Picard-Lindelöf Theorem apply for this example? If so, what is the statement?

   (c) Determine if the Picard approximations converge to the solution of the system.

2. **Use Putzer's Algorithm** to compute $e^{\boldsymbol{A}t}$, where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

3. Fix $k \in \mathbb{R}$, and consider the system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t)$, with $\boldsymbol{A} \in \mathscr{C}^\infty(\mathbb{R}; \mathbb{R}^{2\times 2})$ given by

$$\boldsymbol{A}(t) = \begin{pmatrix} k & 0 \\ \cos(\pi t) & k \end{pmatrix}.$$

   (a) Find the principle matrix solution $\boldsymbol{X} : \mathbb{R} \to \mathbb{R}^{2\times 2}$ at $t = 0$ for this system.

   (b) Identify a monodromy matrix and the Floquet multipliers for the system. Based on the Floquet multipliers, is the **0** solution stable, asymptotically stable, or unstable?

4. Consider the autonomous system

$$\begin{cases} x_1' = x_2 + x_1(x_1^2 + x_2^2)^2 \\ x_2' = -x_1 + x_2(x_1^2 + x_2^2)^2 \end{cases}$$

   (a) Prove that each solution (except $x_1 = x_2 = 0$) of the system blows up in finite time. What is the blow-up time for the solution which starts at the point $(1, 0)$ when $t = 0$?

   (b) Linearize the system around the trivial solution $x_1 = x_2 = 0$.

   (c) Study the stability of the zero solution for the **linearized** system.

Turn to the next page for Part 2.

Math 842 Component of 830/842 Qualifying Exam

Submit work for three of the four problems below.

1. Consider the system

$$x' = x + y, \qquad y' = \alpha x + y,$$

with real parameter α.

   (a) Find the isolated equilibria in terms of α, and classify them as stable, unstable, or saddle; the classification will depend upon α.

   (b) For $\alpha > 1$, draw the nullclines in phase space. In the regions determined by the nullclines, determine the signs of x' and y'. Use these to help you draw the direction arrows on the nullclines, and draw a plausible orbit starting at $(0, 1)$ and a plausible orbit starting at $(1, -1)$.

   (c) In the case where there are equilibria which are not isolated, classify the equilibria as stable, unstable or saddle.

2. Consider the projectile equation

$$\frac{d^2h}{dt^2} = -\frac{R^2 g}{(h+R)^2}, \quad h(0) = 0, \quad \frac{dh}{dt}(0) = V,$$

where h is the distance of the projectile above the surface of the earth, R is the radius of the earth, g is the gravitational constant, and V is the initial upwards velocity.

   (a) Find three dimensionless quantities for this problem in terms of the variables and parameters in the equation.

   (b) Non-dimensionalize the problem so that it becomes

$$\frac{d^2\overline{h}}{dt^2} = -\frac{1}{(1+\varepsilon\overline{h})^2}, \quad \overline{h}(0) = 0, \quad \frac{d\overline{h}}{dt}(0) = 1,$$

   for a parameter ε which is small when Rg is large.

   (c) Use the non-dimensional equation to find an approximate solution which is plausible for falling bodies close to the earth.

3. (a) Use singular perturbation methods to obtain a uniform approximate solution to

$$\epsilon y'' - y' + y^2 = 0, \qquad x \in (0,1), \quad y(0) = -1, \quad y(1) = 0.$$

   You do not need to confirm that the solution is uniformly approximate.

   (b) Find a 3-term approximation to the roots of $(x-2)^2 = \varepsilon x$, for small $\epsilon > 0$.

4. (a) Find all extremals for

$$\int_a^b \left(y(x)^2 + (y'(x))^2 + 2y(x)e^x\right)\, dx.$$

   (b) Fix real numbers y_0 and y_1, and assume that L is a given twice continuously differentiable function on $\mathbb{R}^2$. Let A be the space of all $y \in C^2[a,b]$ with $y(a) = y_0$ and $y(b) = y_1$, and define $J : A \to \mathbb{R}$ by

$$J(y) = \int_a^b L(x, y(x))\, dx.$$

   Derive the Euler equation for this system.

   You should prove this "from scratch", not using any results from the textbook. Be as rigorous as possible, and where you can't be rigorous give plausibility arguments.