Math 830-831. Qualifying exam.
June 2020.

- All problems have equal weight, but their parts, e.g., (a), (b), may be valued differently.
- The parts of a problem are not necessarily related.
- If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.
- Write on one side of the paper and start each new problem on a new page. Hand your work in order.
- **Please be sure to provide full details for solutions. Solutions without adequate justification will not receive full credit.**

Submit work for six of the eight problems below.

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1. **(20 points)** Let f, $u^0 \in C^\infty([0,1])$ be **time-independent** functions which are zero on the boundary, and let $u = u(t,x)$ be a smooth solution to the following problem,
$$\begin{cases} u_t + u^2 u_x = \nu u_{xx} + f, & x \in [0,1],\ t > 0, \\ u(0,x) = u^0(x), & x \in [0,1], \\ u(t,0) = u(t,1) = 0, & t \geq 0. \end{cases}$$
   (a) Assuming $\nu > 0$, show that the energy $\|u\|^2_{L^2([0,1])}$ is bounded in time. [Hint: Use energy methods.]
   (b) If the boundary conditions above were replace by the Neumann conditions $u_x(t,0) = 3$, $u(t,0) = 0$, would the result still hold? Why or why not?

2. **(20 points)** Consider the following traffic model
$$\begin{cases} u_t + (3 - 6u)u_x = 0, & x \in \mathbb{R},\ t > 0, \\ u(0,x) = ax + b, & x \in \mathbb{R}, \end{cases}$$
   where $u(t,x)$ is the (normalized) density of cars at position x and time t, and a, $b \in \mathbb{R}$ are parameters.
   (a) Find the solution of the above system, for all possible values of a and b. Draw the characteristic curves.
   (b) Determine the values of a and b for which shocks develop.
   (c) Identify the flux of this conservation law, interpret the flux physically.

3. **(20 points)** Let $\kappa > 0$. The fundamental solution to the heat equation $u_t = \kappa \triangle u$ on $\mathbb{R}^n$ is given by: $\Phi(t,x) := \frac{1}{(4\pi t\kappa)^{n/2}} e^{-\frac{|x|^2}{4t\kappa}}$ for $t > 0$, and $\Phi(t,x) := 0$ for $t < 0$.
   (a) Let u be a solution to the heat equation with initial data $u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Show that, for $t > 0$, $\|u\|_{L^2} \leq C(\kappa t)^{-r}$ for some constant $C > 0$ and a certain decay rate $r > 0$.
   (b) Let $u = u(t, \mathbf{x})$ be the solution to the heat equation $u_t = \kappa \triangle u$ on $\mathbb{R}^n$ with initial data $u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Show that there exists a constant $C > 0$ such that $\|u(t,\cdot)\|_{L^1} \leq C \|u_0\|_{L^1}$ for all $t > 0$.[Hint: The Fourier transform, $\mathcal{F}[u](\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(\mathbf{x}) e^{-i\xi\cdot\mathbf{x}}\, d\mathbf{x}$ satisfies $\mathcal{F}(e^{-\alpha|\mathbf{x}|^2}) = \frac{1}{(4\pi\alpha)^{n/2}} e^{-\frac{|\xi|^2}{4\alpha}}$ for $\alpha > 0$.]

4. **(20 points)** Let $r > 0$, and denote $B(0,r) := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < r\}$. Let f and g be smooth, bounded functions. Show that, for dimension $n \geq 3$, any solution to
$$\begin{cases} -\triangle u = f & \text{in } B(0,r), \\ \quad u = g & \text{on } \partial B(0,r), \end{cases}$$
   satisfies
$$u(0) = \fint_{\partial B(0,r)} g\, d\sigma + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f\, dx,$$
   where $\alpha(n)$ is the volume of the unit ball $B(0,1)$. [Hint: This is similar to the proof of the mean-value formulas for harmonic functions. It may help to define $\phi(r) := \fint_{\partial B(0,r)} u(\mathbf{y})\, d\sigma(\mathbf{y})$.]

Turn to the next page for Part 2.

5. **(20 points)** Let $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be a smooth function, let $y_0 \in \mathbb{R}$. In parts (a) and (b) below, consider the given methods for approximately solving $y' = f(t, y)$, with $y(0) = y_0$, using step-size $h > 0$.

   (a) Show that the following method has a local truncation error of order 3.

   $$y_{n+1} = y_n + \tfrac{h}{2}\left(f(t_{n+1}, y_{n+1}) + f(t_n, y_n)\right).$$

   Is this a surprising result, or is it expected? Explain.

   (b) **(20 points)** Show that the following method is consistent (you are not asked to find its order). Show also that it is not A-stable.

   $$y_{n+2} = y_{n+1} + h\left(\tfrac{5}{12} f(t_{n+2}, y_{n+2}) + \tfrac{2}{3} f(t_{n+1}, y_{n+1}) - \tfrac{1}{12} f(t_n, y_n)\right).$$

   [Hint: To show it is not A-stable, consider the stability requirements as the step-size increases to infinity.] Is this a surprising result, or is it expected? Explain.

6. **(20 points)**

   (a) Consider the weight function $w(x) = \sqrt{1 - x^2}$, and a polynomial $q(x) = 8x^3 - 4x$. One can check that $\int_{-1}^{1} x^k q(x) w(x)\, dx = 0$ for all $k \in \{0, 1, 2\}$. Find the points x_0, x_1, x_2 (You are not asked to compute the A_i's.) such that the Gaussian quadrature formula

   $$\int_{-1}^{1} f(x) w(x)\, dx = A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2)$$

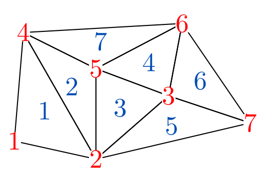
   is exact for all polynomials f of degree at most n, where n is as large as possible. Also state what n is. (You are allowed to use any theorems you know about Gaussian quadrature.)

   (b) Let $x_0 = -1$, $x_1 = 2$. Let $w(x) = 1 + 3x + 3x^2$. (You can assume without proof the fact that $w(x) > 0$ for all $x \in \mathbb{R}$.) Find A_0 and A_1 such that the quadrature formula

   $$\int_{-6}^{6} f(x) w(x)\, dx = A_0 f(x_0) + A_1 f(x_1)$$

   is exact for all polynomials f of degree at most 1.

7. **(20 points)** Consider the following 2D mesh



Suppose that, using piece-wise linear hat-functions $\{\varphi_1, \dots, \varphi_7\}$, a certain "local" matrix is computed on each triangle by: $A_k \begin{pmatrix} 5 & -2 & -2 \\ -2 & 5 & -2 \\ -2 & -2 & 5 \end{pmatrix}$, where A_k is the area of triangle number k. Let $W = (w_{i,j})_{i,j=1}^{7}$ be the global version of this matrix. Compute $w_{1,7}$, $w_{5,2}$, and $w_{6,6}$. Roughly how many operations need to be performed to compute the global stiffness matrix? If you refine the mesh many times, how fast does this number of operations grow in terms of the number of nodes in the mesh?

8. **(20 points)** Let $c > 0$ and $T > 0$ be constants, and let $f \in C_c^\infty(\mathbb{R})$.

   (a) Consider the linear wave equation $u_{tt} = c^2 u_{xx} + f$, posed on the time-space domain $[0, T] \times \mathbb{R}$, with initial data $u(0, \cdot) = u_0 \in C_c^\infty(\mathbb{R})$ and $u_t(0, \cdot) = u_1 \in C_c^\infty(\mathbb{R})$. Derive a formula for the solution to this equation, using the change of variables $\ell = x + ct$ and $r = x - ct$.

   (b) Consider the following nonlinear wave equation $u_{tt} = c^2 u_{xx} - u^3$. Assume that suitable initial and boundary data are given. Propose a numerical scheme to solve this equation in as much detail as possible. Discuss the merits and drawbacks of your scheme in terms of standard concerns of numerical analysis, such as stability, accuracy, implementation, etc. Your discussion should include at least three well-supported points.