# Math 830-831. Qualifying exam.
# May 28, 2019.

- All problems have equal weight, but their parts, e.g., (a), (b), may be valued differently.
- The parts of a problem are not necessarily related.
- If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.
- Write on one side of the paper and start each new problem on a new page. Hand your work in order.

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## Math 830 Component of 830/831 Qualifying Exam

**Submit work for three of the four problems below.**

1. Consider the autonomous system
$$\begin{cases} x_1' = -3x_1 + x_1^2 x_2 \\ x_2' = -x_2 + x_1 x_2 \end{cases}$$
   (a) Find all equilibrium solutions for this system.
   (b) Linearize the system around the trivial solution $x_1 = x_2 = 0$.
   (c) Study the stability of the zero solution for the **linearized** system. Sketch as accurately as possible the phase portrait for the linearized system.

2. Consider the system $\boldsymbol{x}'(t) = \boldsymbol{M}(t)\boldsymbol{x}(t)$ where $\boldsymbol{x} : [0,\infty) \to \mathbb{R}^2$ and $\boldsymbol{M} : [0,\infty) \to \mathbb{R}^{2\times 2}$, with components $m_{i,j}$ continuous and bounded functions on $[0,\infty)$ for $i,j = 1,2$.
   (a) Use any main result from differential equations to show that the system enjoys existence and uniqueness of solutions.
   (b) Use a Gronwall inequality to study the continuity of solutions with respect to initial data.

3. **Use Putzer's Algorithm** to compute the general solution to the system $\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t)$, where
$$\boldsymbol{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

4. Consider the system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t)$, with $\boldsymbol{A} \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^{2\times 2})$ given by
$$\boldsymbol{A}(t) = \begin{pmatrix} 1 + \dfrac{\cos(t)}{2+\sin(t)} & 0 \\ 1 & -1 \end{pmatrix}.$$
   (a) Find the principal matrix solution $\boldsymbol{X} : \mathbb{R} \to \mathbb{R}^{2\times 2}$ at $t = 0$ for this system.
   (b) Identify a monodromy matrix and the Floquet multipliers for the system. Based on the Floquet multipliers, is the $\mathbf{0}$ solution stable, asymptotically stable, or unstable?

***Turn to the next page for Part 2.***

**Math 831 Component of 830/831 Qualifying Exam**

**Submit work for three of the four problems below.**

1. (a) Find the general solution for the equation

$$x^2 u_x + y^2 u_y + z(x+y)u_z = 0.$$

   (b) Find the integral surface for the vector field $V = (-2x, y, 1)$ containing the curve $C : (t, t, t^2)$ for $t > 0$.

2. Consider the traffic model

$$\begin{cases} u_t + (1-2u)u_x = 0, & x \in \mathbb{R},\, t > 0, \\ u(x,0) = ax + b, & x \in \mathbb{R}, \end{cases}$$

   where $u(x,t)$ is the normalized car density at position $x$ and time $t$, and $a$, $b \in \mathbb{R}$ are parameters.

   (a) Find the solution of the above system, in terms of the parameters $a$ and $b$. Draw the characteristic curves.

   (b) Determine the values of $a$ and $b$ for which shocks develop.

   (c) What is the flux of cars per time unit and length unit at a fixed point $(x,t)$? Provide a physical interpretation for the behavior of the system in terms of $a$ and $b$.

3. Consider the damped wave equation

$$u_{tt} - \Delta u + u_t = 0, \quad x \in (0,1),\, t > 0,$$

   with boundary conditions

$$\alpha u_x(0,t) + \beta u_t(0,t) = 0, \quad \gamma u_x(1,t) + \delta u_t(1,t) = 0.$$

   (a) Find values $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ for which one can guarantee that the energy of the system

$$e(t) = \int_0^1 [u_t^2(x,t) + u_x^2(x,t)]\, dx$$

   is decreasing in time.

   (b) Assuming that the energy is decreasing, is it true that the system coupled with initial conditions $(u, u_t)|_{t=0} = (u_0, u_1)$ admits at most one solution?

4. Use the separation of variables method to find an infinite-series representation of the solution to the following heat equation with insulating boundary conditions:

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1,\, t > 0, \\ u_x(0,t) = 0,\ u_x(1,t) = 0, & t > 0, \\ u(x,0) = 1 + 2\cos(3\pi x), & 0 < x < 1. \end{cases}$$