UNL Mathematics Qualifying Exam — January, 1995

850/852

There are 10 questions, of which you should attempt 8. Each question carries equal weight.

All graphs we consider are simple – that is they have no loops or multiple edges. The *girth* of a graph is the length of its shortest cycle (or ∞ if the graph contains no cycles). A *bridge* in a connected graph G is an edge $e \in E(G)$ such that $G \setminus e$ is disconnected. We write $\chi(G)$ for the vertex chromatic number of G; the minimum number of colors with which we can properly color the vertices of G.

Question 1. Prove that a tree with $2k$ endvertices contains k edge disjoint paths connecting the endvertices in pairs.

Question 2. Consider the linear difference operator

$$\mathcal{L}(h_n) = \alpha_k h_{n+k} + \alpha_{k-1} h_{n+k-1} + \ldots + \alpha_0 h_n.$$

Let $p(x) = \alpha_k x^k + \alpha_{k-1} x^{k-1} + \ldots + \alpha_0$ be the associated characteristic polynomial. Show that if $h_n = n\lambda^n$ then $\mathcal{L}(h_n) = n\lambda^n p(\lambda) + \lambda^{n+1} p'(\lambda)$. Hence or otherwise prove that the recurrence relation $\mathcal{L}(h_n) = \lambda^n$ has a solution of the form $h_n = cn\lambda^n$, where c is a constant, if and only if $p(x)$ has a simple root at λ.

Question 3. How many subsets A of $\{1, 2, \ldots, n\}$ are there in which $|i - j| \geq 3$ for all $i, j \in A$, $i \neq j$.

Question 4. State and prove a form of the principle of inclusion/exclusion. Given a positive integer n let $\phi(n) = |\{i \,:\, 1 \leq i \leq n \text{ and } i \text{ is coprime to } n\}|$. Prove that

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

where the product is over the set of all primes dividing n.

Question 5. State and prove Turan's theorem concerning graphs not containing K_r.

Question 6. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ be a family of subsets of a v-set X, each of size k. Assume that each pair $\{x_1, x_2\} \subset X$ is contained in exactly $\lambda > 0$ of the B_i. Prove that if $\lambda < k < v$ then $b \geq v$.

Question 7. Prove that if G is a connected bridgeless planar graph with n vertices, e edges, and girth at least g then

$$e \leq (n-2)\left(\frac{g}{g-2}\right).$$

[Hint: Use Euler's formula.] Prove that neither K_5 nor the Petersen graph is planar.

Question 8. On the island of Combinatorica live n married couples, each consisting of one hunter and one farmer. The Ministry of Hunting divides the island into n hunting ranges of equal area, and, independently, the Ministry of Agriculture divides it into n farming ranges of equal area. The Ministry of Marriage insists that the hunting range and the farming range assigned to each couple must overlap. To everyone's surprise this turns out to be possible. The Ministry of Religion declares it to be a miracle. Assist the Ministry of Atheism in debunking this claim by proving that such an assignment is always possible.

Question 9. Given a directed graph D we define its *span* to be the length of a longest directed path in D. Prove that if G is a k-colourable graph then it can be oriented in such a way that its span is at most k. Show also that if a graph G can be oriented in such a way that it has no directed cycles and its span is at most k then $\chi(G) \leq k$. [Hint: Consider the longest directed path starting from each vertex.]

Question 10. State and prove Burnside's lemma. [You may assume without proof that if a group G acts on a set X then the product, for any element $x \in X$, of the sizes of the stabilizer and the orbit of x is $|G|$.] How many differently coloured cubes can be made using green, red, and blue straws to form the edges? [Hint: The group of roational symmetries of the cube has 24 elements.]