UNL Mathematics Qualifying Exam over 850/852 January, 1995

WORK EXACTLY 8 PROBLEMS. Each question carries equal weight

All graphs we consider are simple - that is they have no loops or multiple edges. A *bridge* in a connected graph G is an edge e of G such that $G - e$ is disconnected. We write $\chi(G)$ for the vertex chromatic number of G; the minimum number of colors with which we can properly color the vertices of G.

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1. Determine the number of ways of selecting r distinct integers out of the first n positive integers such that the selection does not include 2 consecutive integers.

2. Let D_r be the number of derangements of an r-set. Use a combinatorial argument to establish the identity $\sum_{r=0}^{n} \frac{D_r}{r!(n-r)!} = 1$.

3. Prove that if $G = (V, E)$ is a simple graph with $|V| = 2m$, and G has no 3-cycles, then $|E| \leq m^2$.

4. (i) Show that every graph contains two vertices of equal degree.
(ii) Determine all graphs which have exactly one pair of vertices of equal degree.

5. Let G be a connected bipartite graph which is k-regular for some $k \geq 2$. Prove that G is bridgeless.

6. (i) Show that a k-chromatic graph can be oriented in such a way that a longest directed path has at most k vertices.
(ii) Suppose G can be oriented in such a way that no directed path contains more than k vertices and suppose further that G has no directed cycles. Prove that $\chi(G) \leq k$.

7. Consider the recurrence relation $a_n = 3a_{n-1} - 4a_{n-3}$, for $n \geq 3$, where $a_0 = 8, a_1 = (-1), a_2 = (-9)$. Determine an explicit formula for a_n.

8. (i) Prove that if G is a graph in which the degree of every vertex is at least two, then G contains a cycle.
(ii) A tree is a connected acyclic graph. Use (i) to establish that a nontrivial tree contains a vertex of degree one.
(iii) Prove that a graph G with p vertices and q edges is a tree if and only if G is acyclic and $q = p-1$.

9. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ be a family of subsets (called blocks) of a v-set $X = \{x_1, x_2, \ldots, x_v\}$. Further, assume that each unordered pair $\{x_j, x_m\}$, $1 \leq j < m \leq v$, occurs in exactly $\lambda > 0$ blocks. Prove that if $\lambda < |B_i| < v$, for $1 \leq i \leq v$, then $b \geq v$. (Hint: Use an incidence matrix and determinant.)

10. Let G be a permutation group acting on a set X. The *stabiliser* of a point $x \in X$ is $G_x = \{g \in G : gx = x\}$ and the *orbit* of x is $O_x = \{gx : g \in G\}$.
(i) State and prove Burnside's Lemma.
(ii) A cube with edge length 2 is constructed by gluing together 8 cubes with edge length 1. The smaller cubes come in three colors. How many different cubes with edge length 2 can be constructed? (Hint: The group of rotational symmetries of a cube has 24 permutations.)