There are 10 questions, of which you should attempt 8. Each question carries equal weight. Standard results may be quoted provided they are *clearly* stated.

All graphs we consider are simple – that is they have no loops or multiple edges – and finite. A graph is r-*regular* if every vertex has degree r. The union of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. We write $\chi'(G)$ for the chromatic index of G; the minimum number of colours required to properly colour the edges of G.

Question 1. Prove that a sequence of positive integers $(d_i)_1^n$ is the degree sequence for some tree iff each d_i is at least 1 and $\sum_1^n d_i = 2n - 2$. Characterize the degree sequences of forests.

Question 2. How many subsets of $\{1, 2, \ldots, n\}$ contain no pair of adjacent elements?

Question 3. Show that a graph on $2n$ vertices which does not contain a triangle has at most n^2 edges.

Question 4. Define a *selfish* set to be a set A such that $|A| \in A$. How many minimal selfish subsets of $\{1, 2, \ldots, n\}$ are there?

Question 5. A *tournament* is an orientation of K_n for some n. Prove that every tournament contains a directed Hamiltonian path.

Question 6. Suppose that G is a graph with minimal degree $\delta \geq 2$. Prove that G contains a cycle of length at least $\delta + 1$.

Question 7. Let $0 \leq r < n/2$. Let $\binom{\{1,2,\ldots,n\}}{r}$ be the collection of all subsets of $\{1, 2, \ldots, n\}$ of size r. Prove that there is an injection

$$f_r : \binom{\{1, 2, \ldots, n\}}{r} \to \binom{\{1, 2, \ldots, n\}}{r+1}$$

satisfying $A \subset f_r(A)$ for all $A \in \binom{\{1,2,\ldots,n\}}{r}$.

Question 8. Prove that every 4-regular graph, with vertex set V, is the union of two 2-regular graphs on V. [Hint: consider an Euler circuit.] Prove or disprove: every 4-regular graph G has $\chi'(G) = 4$.

Question 9. State and prove Burnside's lemma. [You may assume without proof that if a group G acts on a set X then the product, for any element $x \in X$, of the sizes of the stabilizer and the orbit of x is $|G|$.] How many different ways are there to colour a four by four square grid with 3 colours if rotations (but not reflections) count as the same colouring?

Question 10. Give the general solution to the recurrence

$$h_n - 3h_{n-1} + 4h_{n-3} = 3^n$$