**WORK EXACTLY 8 PROBLEMS. Each question carries equal weight.**

All *graphs* we consider are simple - that is they have no loops or multiple edges.

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1. (a) Show that if a graph is not connected, then its complement is connected.
(b) A graph $G$ has a proper edge coloring with $k$ colors if no two edges of the same color meet at a common vertex. Show that if $G$ is a regular graph with degree 3, where $G$ is hamiltonian, then $G$ has a proper edge coloring with three colors.

2. (a) Define the term "maximal planar graph."
(b) Prove that if $G$ is a maximal planar graph with $p \geq 3$ vertices and $q$ edges, then $q \leq 3p - 6$.
(c) Prove that there exists only one 4-regular maximal planar graph.

3. (a) State and prove a necessary and sufficient condition for a connected graph $G$ to be Eulerian in terms of its degrees.
(b) By finding an Euler circuit in a suitably defined directed graph, construct a circular binary sequence $S$ such that each binary word of length 4 appears exactly once as one moves along the sequence $S$.

4. The cycle double cover conjecture asserts that if $G$ is a connected, bridgeless graph then there exists a multiset $\mathcal{C}$=$\{C_1, C_2, ..., C_m\}$ of cycles of $G$ such that every edge of $G$ is in exactly two cycles from $\mathcal{C}$. [Such a multiset is called a cycle double cover of $G$.] Prove that no graph with a bridge has a cycle double cover. Prove that every planar bridgeless graph has a cycle double cover.

5. Solve the following recurrence relation: $f(n) = 2f(n-1) + f(n-2) - 2f(n-3)$ for $n \geq 3$ where $f(0) = 1, f(1) = 2, f(2) = 0$.

6. A family $\mathcal{F}$ of subsets of $X$ is *intersecting* if $A, B \in \mathcal{F} \Rightarrow \mathcal{A} \cap \mathcal{B} \neq \emptyset$.
(a) Prove that an intersecting family $\mathcal{F}$ of subsets of $X = \{1, ..., n\}$ satisfies $|\mathcal{F}| \leq 2^{n-1}$.
(b) Prove that any intersecting family $\mathcal{F}$ of $X = \{1, ..., n\}$ can be extended to an intersecting family of size $2^{n-1}$.

7. Give combinatorial proofs that a) $D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$ and b) $S(n,k) = kS(n-1,k) + S(n-1,k-1)$, where $D_n$ is the number of derangements of $\{1, 2, ..., n\}$ and $S(n,k)$ is the number of partitions of $\{1, 2, ..., n\}$ into $k$ non-empty subsets.

8. Determine the number of ways of selecting $r$ distinct integers out of the first $n$ positive integers such that the selection does not include 2 consecutive integers.

9. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ be a family of subsets (called blocks) of a $v$-set $X = \{x_1, x_2, \ldots, x_v\}$. Further, assume that each unordered pair $\{x_j, x_m\}$, $1 \leq j < m \leq v$, occurs in exactly $\lambda > 0$ blocks. Prove that if $\lambda < |B_i| < v$, for $1 \leq i \leq v$, then $b \geq v$. (Hint: Use an incidence matrix and determinant.)

10. An $r$ by $n$ Latin rectangle is an $r$ by $n$ matrix where each element from an $n$-set $S$ occurs exactly once in each row and at most once in each column. Prove that an $r$ by $n$ Latin rectangle can be extended to an $(r+1)$ by $n$ Latin rectangle if $r < n$.