There are 10 questions, of' which you should attempt 8. Each question carries equal weight. Standard results may be quoted provided they are *clearly* stated.

We write $[n]$ for the set $\{1, 2, \ldots, n\}$. All graphs we consider are simple – that is they have no loops or multiple edges – and finite. We write $n(G)$ for the number of vertices of G, and $\chi(G)$ for the chromatic number of G (the minimum number of colours required to properly colour the vertices of G).

Question 1. Give combinatorial proofs of the following identities. (Other styles of proof will receive little credit.)

a. $\displaystyle\sum_{i=0}^{k} \binom{n+i}{i} = \binom{n+k+1}{k}$

b. $\displaystyle\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}$

Question 2.

a. Find the general solution to the following recurrence

$$a_n + 5a_{n-1} - 14a_{n-2} = 3^n.$$

b. In order to become a bridge life master (in the American Contract Bridge League) you need a total of 300 points, of which at least 50 must be black, at least 50 silver, and at least 25 must be gold. The only other kind of point available is platinum. Assuming that one can only win whole numbers of each kind of point, how many distributions of 300 points are possible which qualify one as a life master?

Question 3. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ be a family of subsets (called blocks) of $X = [v]$. Further, assume that each unordered pair $\{j, m\} \subset X$ occurs in exactly $\lambda > 0$ blocks. Prove that provided each $i \in [v]$ belongs to strictly more than l of the blocks then $b \geq v$.

Question 4.

a. State and prove the Principle of Inclusion/Exclusion.

b. Give, with justification, a formula for the number of surjections (onto functions) from $[n] = \{1, 2, \ldots, n\}$ to $[k] = \{1, 2, \ldots, k\}$.

Question 5.

a. State Hall's theorem concerning systems of distinct representatives.

b. A *permutation matrix* is $\{0, 1\}$-matrix having exactly one 1 in every row and column. Prove that an $n \times n$ matrix A with non-negative integer entries is a sum of permutation matrices if and only if all the row and column sums are equal.

Question 6. Let T be a tree. We say that T *splits oddly* if for every edge $e \in E(T)$ the two components of $T - e$ each have an odd number of vertices. Prove that the vertices of T all have odd degree if and only if T splits oddly.

Question 7. State and prove Turán's Theorem concerning graphs not containing a copy of K_r.

Question 8.

a. Prove the Szekeres-Wilf Theorem, stating that $\chi(G) \leq 1 + \max_{H \subset G} \delta(H)$.

b. Given a set of n lines in the plane in general position (no two parallel, no three meeting at a point) define a graph on the vertex set consisting of the intersection points of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Prove the that chromatic number of this graph is at most 3.

Question 9.

a. Define a *block* of a graph G, and the *block-cutvertex tree* associated with G.

b. Prove that if G has blocks $B_1, B_2, \ldots, B_k$ then

$$n(G) = 1 - k + \sum_{i=1}^{k} n(B_i).$$

c. Obtain, with justification, a formula for the number of spanning trees of G, given that the number of spanning trees of B_i is t_i.

Question 10.

a. State and prove Burnside's lemma. [You may assume without proof that if a group G acts on a set X then $|\mathrm{Stab}(x)| \cdot |\mathrm{Orb}(x)| = |G|$, where $\mathrm{Stab}(x)$ is the stabilizer of x and $\mathrm{Orb}(x)$ is its orbit.]

b. How many different ways are there to colour the small triangles in the figure below with the colours red, white and blue, if rotations and reflections count as the same colouring?