Do seven of the ten questions. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

All graphs we consider are simple – that is they have no loops or multiple edges – and finite. We denote the chromatic number of a graph G by $\chi(G)$, and its clique number by $\omega(G)$.

Question 1. Let d_n be the number of *derangements* of $[n] = \{1, 2, \ldots, n\}$, that is permutations π of $[n]$ such that $\pi(i) \neq i$ for all $i \in [n]$.

a. Prove that d_n satisfies the recurrence relation

$$d_n = (n-1)(d_{n-1} + d_{n-2}) \qquad \text{for all } n \geq 2.$$

b. By rewriting this recurrence as $d_n - nd_{n-1} = -(d_{n-1} - (n-1)d_{n-2})$, or otherwise, prove that

$$d_n = nd_{n-1} + (-1)^n.$$

c. Prove that d_n is even if and only if n is odd.

Question 2.

a. State and prove Burnside's lemma concerning the number of orbits of a group action.

b. Some identity cards are to be made by taking square cards ruled into a 5×5 grid and punching out two of the squares. How many different cards can be produced this way?

Question 3. Solve the recurrence relation

$$\left.\begin{array}{c} a_n = -4a_{n-1} - a_{n-2} + 6a_{n-3} \\ a_0 = 2, a_1 = -2, a_2 = 10 \end{array}\right\}$$

Question 4.

a. State and prove Hall's theorem concerning the existence of a system of distinct representatives for a family $A_1, A_2, \ldots, A_n$ of sets.

b. Suppose that the A_i satisfy the condition that

$$\left|\bigcup_{i \in S} A_i\right| > |S|$$

for all non-empty $S \subset \{1, 2, \ldots, n\}$. Prove that given any i and any $x \in A_i$ we can find a system of distinct representatives for the A_i in which x represents A_i.

Question 5.
a. State and prove the principle of inclusion/exclusion.
b. You are to make a necklace from n different pairs of beads. The beads in a pair have the same colour but different shapes. In how many ways can you make the necklace so that no two beads of the same colour are adjacent?

Question 6. Let k be an integer with $k \geq 1$, let G be a k-connected graph, and let $S, T \subset V(G)$ be disjoint sets of vertices of G, each of size at least k. Prove that G contains k disjoint S, T paths. (No version of the Fan Lemma may be assumed without proof.)

Question 7. Let $\bar{d} = \bar{d}(G) = 2e(G)/n(G) > 0$ be the average degree of a graph G.
a. Prove that G has a subgraph H with $\delta(H) > \bar{d}/2$. [Hint: consider successively deleting vertices of degree at most $\bar{d}/2$.]
b. Prove that for all $c < 1/2$ there exists a graph G with no subgraph H having

$$\delta(H) > c\bar{d}(G).$$

[Hint: Consider $K_{1,n}$.]

Question 8. Prove that $\chi(G) = \omega(G)$ when $\overline{G}$ is bipartite.

Question 9.
a. State Turán's theorem concerning the maximum number of edges in a graph on n vertices not containing a K_r.
b. Prove that if G is a graph with $n \geq r+1$ vertices and $t_{r-1}(n)+1$ edges then for every n' with $r \leq n' \leq n$ there is a subgraph H of G with n' vertices and at least $t_{r-1}(n')+1$ edges. [Hint: consider a vertex in G of minimum degree.]
c. From the previous part deduce Turán's theorem, and also the stronger fact that such a G contains two K_r subgraphs sharing $r-1$ vertices.

Question 10. Consider a connected plane graph G with dual G^*. Prove that a subset of $E(G)$ forms a spanning tree if and only if the duals of the remaining edges form a spanning tree of G^*.