Do seven of the ten questions. Of these at least three should be from section A and at least three from section B. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Section A.

Question 1.

a. Give a combinatorial proof of the following identity:

$$\binom{n}{k} = \sum_{i=k-1}^{n-1} \binom{i}{k-1}.$$

[Proofs by other techniques will receive little credit.]

b. Find a closed form for the expression $\displaystyle\sum_{i=1}^{n} \frac{i}{n^i}\binom{n}{i}$.

Question 2.

a. State and prove the principle of inclusion/exclusion.

b. In a small town n married couples attend a town meeting, and each of these $2n$ people want to speak exactly once. In how many ways can the speakers be scheduled if we insist that no married couple speaks in consecutive slots?

Question 3.

a. Prove that if P is a poset such that no chain has length more than k then P can be written as the union of at most k antichains. [Hint: This is not Dilworth's theorem.]

b. Prove that a sequence of distinct real numbers of length n^2 must contain either an increasing subsequence of length $n+1$ or a decreasing subsequence of length $n+1$.

Question 4.

a. Let C be a binary linear code, and let E be the subset of C consisting of those codewords having even length. Prove that $|E|$ is either $|C|$ or $|C|/2$.

b. Prove that given positive integers n, q, d there is a q-ary code of length n and minimum distance d having at least

$$\frac{q^n}{\sum_{i=0}^{d-1}\binom{n}{i}(q-1)^i}$$

codewords. [Note: we are not requiring this code to be linear.]

Question 5.

a. State and prove Burnside's lemma concerning the number of orbits of a group action. [You may assume without proof that if a group G acts on a set X then $|\mathrm{Stab}(x)| \cdot |\mathrm{Orb}(x)| = |G|$, where $\mathrm{Stab}(x)$ is the stabilizer of $x \in X$ and $\mathrm{Orb}(x)$ is its orbit.]

b. How many ways are there to make a 9 bead necklace out of red, white, and black beads if two necklaces which are rotations of each other are considered to be the same, but necklaces which are reflections of one another are considered distinct?

Section B.

Question 6. Recall that a *trail* is a walk in a graph that does not use any edge more than once. A graph is called *randomly Eulerian from vertex* x if every maximal trail starting at x is an Euler circuit. Prove that G is randomly Eulerian starting at x iff G has an Euler circuit and x is contained in every cycle of G.

Question 7. Let m and n be positive integers, and set $X = \{1, 2, \dots, mn\}$. Suppose that X is partitioned into m sets $A_1, A_2, \dots, A_m$ each of size n, and also independently partitioned into m sets $B_1, B_2, \dots, B_m$ each of size n. Prove that one can renumber the sets B_i so that $A_i \cap B_i \neq \emptyset$ for $i = 1, 2, \dots, m$.

Question 8. Let G be an edge-maximal graph on n vertices not containing a K_r, and suppose that $n \geq r + 1$. Let $T_{r-1}(n)$ be the complete $(r-1)$-partite graph on n vertices whose class sizes are as equal as possible, and let $t_{r-1}(n) = e(T_{r-1}(n))$.
a. Prove that G contains a K_{r-1} on some subset $A \subset V(G)$.
b. Prove that $e(G) \leq \binom{r-1}{2} + (r-2)(n-r+1) + e(G \setminus A)$ and deduce that $e(G) \leq t_{r-1}(n)$.
c. Prove that a graph on n vertices having $t_{r-1}(n)$ edges and not containing a K_r is $T_{r-1}(n)$.

Question 9. An *interval graph* is a graph whose vertices consist of closed intervals in $\mathbb{R}$, and where two intervals are adjacent if they are not disjoint. Prove that if G is an interval graph then $\chi(G) = \omega(G)$. [Hint: consider using the greedy algorithm.]

Question 10.
a. Let G be a graph on $n \geq 3$ vertices such that for all pairs of non-adjacent vertices x, y we have $d(x) + d(y) \geq n$. Prove that G has a Hamilton cycle. [Hint: consider a longest path in G.]
b. Prove that the result in part **a** is best possible, by constructing a graph for every $n \geq 3$, having n vertices and satisfying $d(x) + d(y) \geq n - 1$ for all pairs of non-adjacent vertices, which does not have a Hamilton cycle.