Master's/Qualifying Exam over 850/852 June 5, 1992

WORK EXACTLY 8 PROBLEMS.

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1. If graph G has cycles, then the length of the shortestcycle in G is called the girth of G.
(a) Prove that if G is a connected planar graph with p vertices, q edges, and girth k, then $q \leq (p-2)(\frac{k}{k-2})$.
(b) Use (a) to prove that $K(3,3)$ is nonplanar.
(c) Use (a) to prove that the Petersen graph (the unique (3,5)-cage) is nonplanar.

2. Let G be a simple graph with p vertices, q edges, and k components. Prove:
(a) $q \geq (p-k)$, with equality if and only if every component of G is a tree.
(b) G has at least $q-p+k$ distinct cycles. (Cycles are distinct if and only if they have different edge sets).

3. (a) Show that in any graph G, a vertex of odd degree must always be connected by a path to another vertex of odd degree.
(b) Prove: If a simple graph G has two vertices that are not connected by a path of length 3 or less, then every pair of vertices in $\bar{G}$ are connected by a path of length 3 or less.

4. (a) State P. Hall's theorem on the existence of a system of distinct representatives.
(b) Using (a) (or an equivalent theorem) prove the following: If G is a finite bipartite graph which is regular of positive degree, then G has a perfect matching.

5. (a) Define: G is a tournament graph.
(b) Prove that every tournament contains a hamiltonian path.

6. Let S be a finite set of objects and let A_i be the subset of S consisting of those objects possessing property P_i for $1 \leq i \leq m$.
(a) Prove the following form of the inclusion-exclusion principle: The number of objects of S which have at least one of the properties $P_1, P_2, \ldots, P_m$ is given by
$|A_1 \cup A_2 \cup \cdots \cup A_m| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \cdots + (-1)^{m+1} |A_1 \cap A_2 \cap \cdots \cap A_m|$,
where each sum is taken over the appropriate possible subsets of indices.
(b) Use (a) to prove: If n is a positive integer whose prime factorization is $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, then the number of integers from 1 to n (inclusive) that are relatively prime to n is the function $\phi(n) = n(1-\frac{1}{p_1})(1-\frac{1}{p_2}) \cdots (1-\frac{1}{p_k})$.

7. (a)Establish a recurrence relation for $f(n)$, where $f(n)$ is the number of regions created by n mutually intersecting planes in3-dimensional space such that every three planes meet in one point, butno four planes have a point in common.
(b) Establish a recurrence relation for $g(n)$, where $g(n)$ is the number of ways to pair off with nonintersecting lines $2n$ different points on a circle.

8. Solve the following recurrence relation: $f(n) = 3f(n-2) - 2f(n-3)$ for $n \geq 3$ where $f(0) = 2, f(1) = -2, f(2) = 21$. (Note the order of this recurrence).

9. Let B $= \{B_1, B_2, \ldots, B_b\}, |B_i| = k$ for $1 \leq i \leq b$, be a family of subsets (called blocks) of a v-set $X = \{x_1, x_2, \ldots, x_v\}$. Further, assume that each unordered pair $\{x_j, x_m\}$,

$1 \leq j < m \leq v$,occurs in exactly $\lambda > 0$ blocks.
(a) Prove that each element of X occurs the same number of times r among the blocks.
(b) Prove that if $k < v$, then $b \geq v$. (Hint: Use an incidence matrix and determinant).

10. Prove the formula that counts the number of solutions $(x_1, x_2, \ldots, x_k)$, with x_i nonnegative integers, $1 \leq i \leq k$, to $x_1 + x_2 + \cdots + x_k = r$.