Master's/Qualifying Exam over 850/852 June 3, 1993

WORK EXACTLY 8 PROBLEMS.

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1. Let G be a simple undirected graph with p vertices, q edges, and k components. Assume that G is embedded in a plane and partitions the plane into f regions (faces).
(a) Prove the formula $p - q + f = k + 1$.
(b) Assuming the formula in (a), prove that G has a vertex of degree ≤ 5.

2. (a) State P. Hall's theorem on the existence of a system of distinct representatives.
(b) An r by n Latin rectangle is an r by n matrix where each element from an n-set S occurs exactly once in each row and at most once in each column. Prove that an r by n rectangle can be extended to an $(r+1)$ by n Latin rectangle if $r < n$.

3. Let B be the set of triples, or blocks, in a Steiner triple system on a set X of v points. By definition, B is a collection of 3-subsets of X such that every 2-subset of X is contained in exactly one of the triples (3-subsets). A graph G is formed with B as vertex set, where two elements of B are adjacent as vertices of G if and only if they intersect as subsets of X.
(a) Show that G is regular (each vertex has the same degree) and express its degree in terms of v.
(b) Show that every edge of G is contained in a uniform number of triangles of G, and express this number in terms of v.

4. A directed graph is called strongly connected if there is a path from x to y for any two vertices x, y in G. Prove that G is strongly connected if and only if G's vertices cannot be partitioned into two sets V_1, V_2 such that there are no edges from a vertex in V_1 to a vertex in V_2.

5. (a) State and prove the principle of inclusion and exclusion.
(b) Let D_n be the number of derangements of (1,2,...,n), that is, permutations which leave no element fixed. Prove that $D_n = n! \sum_{r=0}^{n} (-1)^r \frac{1}{r!}$.

6. Let B $= \{B_1, B_2, \ldots, B_b\}, |B_i| = k$ for $1 \leq i \leq b$, be a family of subsets (called blocks) of a v-set $X = \{x_1, x_2, \ldots, x_v\}$. Further, assume that each unordered pair $\{x_j, x_m\}$, $1 \leq j < m \leq v$, occurs in exactly $\lambda > 0$ blocks.
(a) Prove that each element of X occurs the same number of times r among the blocks.
(b) Prove that if $k < v$, then $b \geq v$. (Hint: Use an incidence matrix and determinant).

7. (a) Prove the formula that counts the number of solutions $(x_1, x_2, \ldots, x_k)$, with x_i nonnegative integers, $1 \leq i \leq k$, to $x_1 + x_2 + \cdots + x_k = r$.
(b) How many solutions are there to $x_1 + x_2 + \cdots + x_k = r$, where $x_i \geq i, i = 1, ..., k$?

8. Solve the following recurrence relation: $f(n) = 5f(n-1) - 3f(n-2) - 9f(n-3)$ for $n \geq 3$ where $f(0) = 1, f(1) = 7, f(2) = 65$.

9. (a) State and prove a necessary and sufficient condition for a connected graph G to be Eulerian in terms of its degrees.
(b) By finding an Euler circuit in a suitably defined directed graph,construct a circular binary sequence S such that each binary word of length 4 appears exactly once as one moves along the sequence S.

10. A Hadamard matrix is a square n by n matrix of ± 1's such that $HH^t = nI$.
(a) Show that $H^t H = HH^t$.
(b) If $n > 2$ show that 4 divides n by examining the inner products between the first three rows of H.