UNL Mathematics Qualifying Exam **June, 1994**
850/852

There are nine questions, of which you should attempt seven. Each question carries equal weight.

All graphs we consider are simple – that is they have no loops or multiple edges. A *bridge* in a connected graph G is an edge e of G such that $G - e$ is disconnected. We write $\chi(G)$ for the vertex chromatic number of G; the minimum number of colors with which we can properly color the vertices of G.

Question 1. Let G be a permutation group acting on a set X. The *stabiliser* of a point $x \in X$ is $G_x = \{g \in G : gx = x\}$ and the *orbit* of x is $O_x = \{gx : g \in G\}$. State and prove Burnside's Lemma. [You may assume that $|O_x| = |G|/|G_x|$.] Now consider the diagram below. How many ways are there to color the 16 small triangles with 3 colors if rotations and reflections are considered the same coloring? [Hint: Use Burnside's Lemma.]

Question 2. In a nursery school, m children take off their wet socks and put them on the radiator to dry. At the end of the day, after the socks have become hopelessly muddled, each child takes two socks. Use the principle of Inclusion/Exclusion to calculate the number of ways in which each child gets at least one sock that isn't his or hers, and also the number of ways in which all the socks find themselves on the wrong feet [Little Johnny's socks are of course indistinguishable from one another, but distinguishable from anyone else's socks.]

Question 3. Give combinatorial proofs that

a) $d_n = (n-1)d_{n-1} + (n-1)d_{n-2}$ b) $S(n,k) = kS(n-1,k) + S(n-1,k-1)$.

where d_n is the number of derangements of $\{1, 2, \ldots, n\}$, i.e., $d_n = |\{\pi \in \Sigma_n : \pi(i) \neq i \ \forall i\}|$, and $S(n,k)$ is the number of partitions of $\{1, 2, \ldots, n\}$ into k non-empty subsets.

Question 4.

a) In the old Sussex game of Six-Man's-Noodle six dice are rolled and the player wins if at least one 1 is thrown. In the variant Twelve-Man's-Noodle 12 dice are thrown and to win you need at least two 2s. Which game gives you a better chance of winning?

b) A *four-flush* in poker is a five card hand with exactly four cards all of one suit. How many poker hands are a four-flush and no better? [Relevant poker information: a four-flush beats a pair but is beaten by a flush (five cards of the same suit) or a straight (5 cards in numerical sequence independent of suit).]

c) M^c^Wendy's serves four different Breakfast Specials. How many different orders of six Breakfast Specials are there?

Question 5.

a) Using the method of generating functions, solve the recurrence relation

$$a_n = 2a_{n-1} - 5a_{n-2},\ n \geq 2 \qquad a_0 = -1,\ a_1 = 0.$$

b) How many ways are there to distribute eight different toys to four children if the first child must get at least two?

c) How many ways are there to paint ten identical chairs with the colors red, yellow, green, and blue, if there is only enough red and blue for 4 chairs each. [There's plenty of green and yellow paint.]

Question 6. Let G be a graph with degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n$, with vertex v_i having degree d_i. Prove that we can order the vertices in such a way that the greedy algorithm for coloring G uses no more that $\max_{i \leq n} \min(d_i + 1, i)$ colors. Deduce that if k is $\max\{i \leq n : i \leq d_i + 1\}$ then $\chi(G) \leq k$. Deduce that if G' is the complement of G then $\chi(G) + \chi(G') \leq n + 1$.

Question 7.

a) Let G, H be two vertex disjoint graphs with $e = x_1x_2$ an edge of G and $f = y_1y_2$ an edge of H. Denote by $G\Delta H$ the graph obtained from the union of G and H by deleting edges e and f, adding a new edge x_2y_2, and identifying vertices x_1 and x_2. Prove that if $\chi(G) \geq k$ and $\chi(H) \geq k$ then $\chi(G\Delta H) \geq k$.

b) Prove that if G is a graph with $\chi(G) \leq k$ then the edges of G can be oriented in such a way that no directed path has length greater than k.

Question 8.

a) Let G be a connected bipartite graph which is k-regular for some $k \geq 2$. Prove that G is bridgeless.

b) Let G be a k-regular graph with n vertices which contains a K_r for some $r \geq \lfloor n/2 \rfloor + 1$. Prove that $G = K_n$.

Question 9. The cycle double cover conjecture asserts that if G is a connected, bridgeless graph then there exists a multiset $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of cycles of G such that every edge of G is in exactly two cycles from $\mathcal{C}$. [Such a multiset is called a cycle double cover of G.] Prove that no graph with a bridge has a cycle double cover. Prove that every planar bridgeless graph has a cycle double cover.