UNL Mathematics Qualifying Exam **June, 1996**
850/852

There are 10 questions, of which you should attempt 8. Each question carries equal weight.

All graphs we consider are simple – that is they have no loops or multiple edges. An *independent set* in a graph is a subset of the vertices, not two of which are joined by an edge. A graph is *k-partite* if its vertex set can be partitioned into k independent sets.

Question 1. Prove that if G is a graph with $e(G) \geq n(G)$ then G contains a cycle. Now let $A_1, A_2, \ldots, A_n$ be n distinct subsets of $\{1, 2, \ldots, n\}$. Prove that there exists some $i \in \{1, 2, \ldots, n\}$ such that the sets $A_j \setminus \{i\}$, $j = 1, 2, \ldots, n$ are all distinct.

Question 2. State and prove Hall's Theorem concerning matchings in bipartite graphs. [No result essentially equivalent to Hall's theorem may be assumed.] Let G be a bipartite graph with bipartition $\{X, Y\}$ and suppose that $d(x) = k$ for all $x \in X$ and $d(y) = l$ for all $y \in Y$. Prove that G contains a matching of the smaller of X and Y into the larger.

Question 3. Solve the following recurrence relation:

$$x_n - x_{n-1} - 5x_{n-2} - 3x_{n-3} = 0$$
$$x_0 = 1, x_1 = 2, x_3 = 19$$

Question 4. State and prove an explicit formula for the number of integer solutions to the equation $\sum_{i=1}^{r} x_i = n$ subject to $1 \leq x_i \leq a_i$.

Question 5. Prove that if G is a graph with n vertices and maximum degree Δ then G contains an independent set of size at least $n/(\Delta + 1)$.

Question 6. The Catalan numbers, $(C_i)_{i=1}^{\infty}$, satisfy the recurrence

$$C_1 = 1, \qquad C_n = \sum_{i=1}^{n-1} C_i C_{n-i}.$$

Let $F(t) = \sum_{i=1}^{\infty} C_i t^i$ be the ordinary power series generating function of the sequence of Catalan numbers. Prove that $F(t) = t + (F(t))^2$, and hence (or otherwise) derive explicit expressions for $F(t)$ and C_n.

Question 7. State and prove Burnside's lemma. [You may assume without proof that if a group G acts on a set X then the product, for any element $x \in X$, of the sizes of the stabilizer and the orbit of x is $|G|$.] How many differently coloured octahedra can be made from balls and sticks if the balls available come in three different colours? [Hint: The group of rotational symmetries of the octahedron has 24 elements.]

Question 8. Suppose G is a graph with $n(G) = 2m+1$ such that G does not contain a 3-cycle. Prove that $e(G) \leq m^2 + m$.

Question 9. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ with the property that each v_i is adjacent to at most $k-1$ of the vertices $v_1, v_2, \ldots, v_{i-1}$. Show that G is k-partite.

Question 10. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ be a family of subsets of a v-set X, each of size k. Assume that each pair $\{x_1, x_2\} \subset X$ is contained in exactly $\lambda > 0$ of the B_i. Prove that if $\lambda < k < v$ then $b \geq v$.