There are 10 questions, of which you should attempt 8. Each question carries equal weight. Standard results may be quoted provided they are *clearly* stated.

All graphs we consider are simple – that is they have no loops or multiple edges – and finite. A graph is *r-regular* if every vertex has degree r. We write $\chi(G)$ for the chromatic number of G (the minimum number of colours required to properly colour the vertices of G) and $\chi'(G)$ for the chromatic index of G (the the minimum number of colours required to properly colour the edges of G). The union of two graphs (V_1, E_1) and (V_2, E_2) is the graph $(V_1 \cup V_2, E_1 \cup E_2)$.

Question 1. How many solutions in integers are there to the system

$$\left.\begin{array}{r} x_1, x_2, \ldots, x_k \geq 1 \\ x_1 + x_2 + \ldots + x_k = m \end{array}\right\}?$$

Prove your answer. How many subsets of $\{1, 2, \ldots, n\}$ contain no subset of the form $\{i, i+1\}$, $1 \leq i \leq n-1$?

Question 2. Let S be a set of points in the plane such that $|x - y| \geq 1$ for all $x, y \in S$, $x \neq y$. If $|S| = n$, show that at most $3n - 6$ pairs of points in S have $|x - y| = 1$.

Question 3. Find the general solution to the following recurrence

$$a_n + 3a_{n-1} - 10a_{n-2} = 2^n.$$

Question 4. Suppose that $\mathcal{A} = (A_i)_1^n$ is a family of subsets of $\{1, 2, \ldots, n\}$. Define an $n \times n$ matrix M by

$$M = (m_{ij})_{i,j=1}^n \qquad m_{ij} = \begin{cases} 1 & i \in A_j \\ 0 & i \notin A_j \end{cases}.$$

Show that if M is invertible then the family $\mathcal{A}$ has a system of distinct representatives.

Question 5. Prove that if G is a 3-regular graph with a Hamilton cycle then $\chi'(G) = 3$.

Question 6. State and prove Burnside's lemma. [You may assume without proof that if a group G acts on a set X then $|\mathrm{Stab}(x)| \cdot |\mathrm{Orb}(x)| = |G|$, where $\mathrm{Stab}(x)$ is the stabilizer of x and $\mathrm{Orb}(x)$ is its orbit.] How many different ways are there to colour a five by five square grid with 2 colours if rotations (but not reflections) count as the same colouring?

Question 7. Suppose that $n = 2^p + 1$ for some integer p. Prove, by considering the chromatic number or otherwise, that K_n cannot be written as the union of p bipartite graphs.

Question 8. State and prove the Principle of Inclusion-Exclusion. Use it to prove that if we define

$$\phi(n) = |\{i \,:\, 1 \leq i \leq n, (i, n) = 1\}|$$

then if $p_1, p_2, \ldots, p_k$ are the prime divisors of n we have

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_k}\right).$$

Question 9. Prove that if G is a graph with minimum degree at least 2 then G contains a cycle. Deduce that every tree (a connected acyclic graph) contains a vertex of degree exactly 1. Prove that a graph with n vertices and e edges is a tree if and only if it is acyclic and $e = n - 1$.

Question 10. Prove that a graph on $2n$ vertices which does no contain a triangle has at most n^2 edges.